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On tensor product approximation of analytic functions

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Abstract

We prove sharp, two-sided bounds on sums of the form $\sum_{\boldsymbol{k}\in\mathbb{N}_0^d\setminus\mathcal{D}_{\boldsymbol{a}}(T)} \exp(-\sum_{j=1}^d a_j k_j)$, where $\mathcal{D}_{\boldsymbol{a}}(T) := \{\boldsymbol{k} \in \mathbb{N}_0^d : \sum_{j=1}^d a_j k_j \le T\}$ and $\boldsymbol{a} \in \mathbb{R}_+^d$. These sums appear in the error analysis of tensor product approximation, interpolation and integration of d-variate analytic functions. Examples are tensor products of univariate Fourier–Legendre expansions (Beck et al., 2014) or interpolation and integration rules at Leja points (Chkifa et al., 2013), (Narayan and Jakeman, 2014), (Nobile et al., 2014). Moreover, we discuss the limit $d \to \infty$, where we prove both, algebraic and sub-exponential upper bounds. As an application we consider tensor products of Hardy spaces, where we study convergence rates of a certain truncated Taylor series, as well as of interpolation and integration using Leja points.

Keywords: Multivariate approximation; Interpolation; Integration; Sparse grids; Infinite dimensions; Analytic functions

1. Introduction

Recently, the approximation, interpolation and integration of analytic functions have drawn a lot of interest, especially in the area of uncertainty quantification [3,4,17,29]. Among the most popular approaches are generalized sparse grids, which use tensor products of certain univariate approximation schemes, like orthogonal polynomial expansions [6,16,23], Tschebyscheff

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interpolation [41,42], Gaussian and Clenshaw–Curtis quadrature [41,42], Taylor expansions [16,17,55] or interpolation at Leja points [13,14,40,41]. For integration problems also special quasi-Monte Carlo methods have been developed [21,22], which are able to achieve algebraic rates of convergence $\mathcal{O}(N^{-r})$ of arbitrarily high order.[1] Moreover, there exist hybrid methods, which allow unstructured point sets to project analytic functions onto a tensor product basis by a least squares fitting approach [15,39]. Finally, there are kernel based methods which use orthogonal projections onto a certain basis of a given reproducing kernel Hilbert space of smooth functions, see e.g. [29,46].

In this paper, we will study approximation algorithms that employ sparse tensor products of univariate approximation schemes which on the one hand allow for exponential convergence and on the other hand are maximally nested, i.e. on each level only one additional function(al) evaluation is needed. This differs from classical approaches like, e.g., Clenshaw–Curtis quadrature [27,43] or piecewise linear splines [10], where the number of point evaluations usually doubles from level to level. The associated sparse grid or Smolyak methods were originally tailored to function spaces with dominating, but finite mixed smoothness, e.g. H^r_{mix}. They have been thoroughly analyzed in this setting [10,24,49,57] and sharp upper and lower bounds are available. However, analytic tensor product spaces and their approximability properties are not that well understood yet, albeit there has been steady progress [5,6,33,34,37,42,45,55].

To this end, we consider the general problem of approximating a bounded linear operator $I_d : \mathcal{H}^{(d)} \to \mathcal{G}$ between the d-fold tensor product of Banach spaces[2] of univariate analytic functions $\mathcal{H}^{(d)} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_d$ and a normed linear space $\mathcal{G}$. Often, I_d is also referred to as *solution operator*, see e.g. [56]. We assume that I_d has a representation as an infinite series, i.e.

$$I_d(f) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \Delta_{\boldsymbol{k}}(f), \tag{1.1}$$

where $\Delta_{\boldsymbol{k}} : \mathcal{H}^{(d)} \to \mathcal{G}$ is also bounded and linear and requires the evaluation of exactly one (additional) linear functional $L_{\boldsymbol{k}} : \mathcal{H}^{(d)} \to \mathbb{R}$, i.e. $\Delta_{\boldsymbol{k}}(f) = L_{\boldsymbol{k}}(f)\varphi_{\boldsymbol{k}}$, where $\varphi_{\boldsymbol{k}} \in \mathcal{G}$.

It is natural to discretize I_d by truncating the series (1.1), i.e.

$$\mathcal{A}_T(f) := \sum_{\boldsymbol{k}\in\mathcal{F}(T)} \Delta_{\boldsymbol{k}}(f) \approx I_d(f), \tag{1.2}$$

where $\mathcal{F}(T) \subset \mathbb{N}_0^d$ is a finite index set parametrized by $T \in \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$, which exhausts the whole $\mathbb{N}_0^d$ as $T \to \infty$ and fulfills the conditions[3]

$$\boldsymbol{k} \leq \boldsymbol{v} \wedge \boldsymbol{v} \in \mathcal{F}(T) \Rightarrow \boldsymbol{k} \in \mathcal{F}(T). \tag{1.3}$$

This means that $\mathcal{F}(T)$ has no holes and that the approximation algorithm $\mathcal{A}_T$ converges for every $f \in \mathcal{H}^{(d)}$ to I_d, as T tends to infinity. Then, the error of $\mathcal{A}_T$ can be bounded through

$$\|I_d(f) - \mathcal{A}_T(f)\|_{\mathcal{G}} \leq \sum_{\boldsymbol{k}\in\mathbb{N}_0^d\setminus\mathcal{F}(T)} \|\Delta_{\boldsymbol{k}}(f)\|_{\mathcal{G}} \leq \sum_{\boldsymbol{k}\in\mathbb{N}_0^d\setminus\mathcal{F}(T)} \|\Delta_{\boldsymbol{k}}\|_{\mathcal{H}^{(d)}\to\mathcal{G}} \|f\|_{\mathcal{H}^{(d)}}, \tag{1.4}$$

[1] Recently, also super-algebraic rates of convergence have been proven for analytic functions in [51].

[2] Which has to be equipped with a suitable crossnorm, see e.g. [30].

[3] The notation $\boldsymbol{k} \leq \boldsymbol{v}$ is short for the component-wise relation $k_j \leq v_j$ for all $j \in \{1, \ldots, d\}$.