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The closure in a Hilbert space of a preHilbert space Chebyshev set that fails to be a Chebyshev set

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Abstract

In 1987 the author gave an example of a non convex Chebyshev set S in the incomplete inner product space E consisting of the vectors in l_2 which have at most a finite number of non zero terms. In this paper, we show that the closure of S in the Hilbert space completion l_2 of E is not Chebyshev in l_2.

Keywords: Convex; Unique nearest point; Euclidean space; Hilbert space approximation; Chebyshev set

1. Introduction

Recall that a set C in a normed linear space X is called a *Chebyshev set* if each point in X has a unique nearest point in C. It is well known (and goes back at least to Riesz [12] in 1934) that every closed convex subset of a Hilbert space is a Chebyshev set. A natural question is the converse: must every Chebyshev subset of a Hilbert space be convex? Bunt [2] showed in his 1934 Dutch doctoral thesis that in finite dimensional Hilbert spaces, every Chebyshev set is convex. In 1965 Klee [11] conjectured that a non convex Chebyshev set must exist in some infinite dimensional Hilbert space and gave some evidence for this conjecture. See also Klee [10]. (A more complete historical description of most results related to the convexity of Chebyshev sets 'COCS' problem can be found in survey [3].)

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In [8] it was shown that the construction given in [7] could be modified such that the non convex set S is bounded. Other results associated with the set S can be found in [4,5,9].

Asplund [1] and Vlasov [13] have other results related to nonconvex Chebyshev sets. In [6] there is an abstraction of the notion of convexity in metric spaces.

The supposition that every point in H has a unique nearest point in $\overline{S}$, the closure of S, leads to a contradiction. Using the bounded set S described in [8] we shall show that if we assume that every point in H has a unique nearest point in $\overline{S}$, in H then there is a particular point v in H that does not have a unique nearest point in $\overline{S}$.

In [7,8] sequence of surfaces $S_1, S_2, \ldots$ were constructed such that $S_n \subset S_{n+1}$ and $\bigcup S_n = S$. Each S_n has a unique 'lowest point' and with each such lowest point we associate a unique point v_n. The step by step construction creates a sequence $v_1, v_2, \ldots$ of points, associated with these 'lowpoints', that converges in H to a point v. We shall suppose that v has a unique nearest point u in $\overline{S} \subset H$.

It will then be shown that there is point w distinct from u in $\overline{S}$, such that $\|w - v\| \mid \leq \|u - v\|$, hence v does not have a unique nearest point in $\overline{S}$.

2. Background and notation

A set S was constructed in [7] as a subset of the real inner product space E of all real sequences having at most a finite number of nonzero terms, with inner product $(x, y) = \sum_i x_i y_i$, where $x = (x_1, x_2, \ldots),\ y = (y_1, y_2, \ldots)$ and induced norm $\|x\| = \sqrt{(x, x)}$.

The standard orthonormal basis for E is $\{\phi_1, \phi_2, \ldots\}$ where for each positive integer n, ϕ_n is that sequence in E for which each term is zero except the nth term, which is one.

We follow some of the notation given in [7,8] and note that it was shown there that there is a sequence of functions $\{F_n\}_{n=0}^{\infty}$ and a positive number sequence $\{A_i\}_{i=0}^{\infty}$ that converges to 0, called a determining sequence, used to define S.

It was shown in [8] that if the additional constraint, for each positive integer n, $0 < A_n \leq (8/9)[(3/2)^{1/2^n} - 1]$, is placed on the determining sequence $\{A_i\}_{i=0}^{\infty}$, then the resulting set S is bounded and the core set C, is precompact.

It is this set S, that is of interest for this work as well as the set C, whose closure $\overline{C}$ in H, as was shown in [8], is a compact set. Also note that no line intersects S three times.
The following definitions are from [7]:

$$
\begin{aligned}
&a_0 = 2, \qquad A_0 = 1, \qquad F_0 = 1, \qquad L_0 = 1,\\
&d_1 = \{x_1 : -F_0 \leq x_1 \leq a_0 F_0\},\\
&D_1 = \{x_1\phi_1 : x_1 \in d_1\},\\
&h_1(x) = x_1 : x \in D_1,\\
&L_1(x_1) = a_0 F_0^2 + (a_0 - 1)F_0 x_1 - x_1^2 : x_1 \in d_1,\\
&F_1^2(x_1) = 2L_1(x_1)/[a_0 + 1] : x_1 \in d_1,\\
&S_1 = \{x_1\phi_1 - F_1(x_1)\phi_2 : x_1 \in d_1\},\\
&g_{1,1}(x_1) = x_1 + 1/2\big(F_1^2\big)'(x_1) = [(a_0 - 1)F_0 - 2x_1]/[a_0 + 1] : x_1 \in d_1,\\
&G_1(x) = g_{1,1}(h_1(x))\phi_1 : x \in D_1
\end{aligned}
$$

and C_1 is the image of D_1 under G_1.
We use the following notation:

$$x_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,n}),$$