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Full length article

Density of certain polynomial modules

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Abstract

In this paper the problem of density in the space $C(X)$, for a compact set $X \subset \mathbb{C}$, of polynomial modules of the type $\{p + \bar{z}^d q \colon p, q \in \mathbb{C}[z]\}$ for integer $d > 1$, as well as several related problems are studied. We obtain approximability criteria for Carathéodory compact sets using the concept of a d-Nevanlinna domain, which is a new special analytic characteristic of planar simply connected domains. In connection with this concept we study the problem of taking roots in the model spaces, that is, in the subspaces of the Hardy space H^2 which are invariant under the backward shift operator.

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1. Introduction

Through this paper let X be a compact set in the complex plane $\mathbb{C}$. Denote by $\mathcal{P}$ the space of all polynomials in the complex variable, by $\mathcal{R}(X)$ the space of all rational functions in the complex variable having their poles outside X, as well as by $C(X)$ the space of all continuous complex-valued functions on X endowed with the uniform norm $\|f\|_X = \max_{z\in X}|f(z)|$. Furthermore, let z and $\overline{z}$ denote the complex variable and its conjugation as well as the identity function and the function $z \mapsto \overline{z}$ respectively.

Take an integer $d \geqslant 1$ and define the following spaces of functions

$$\mathcal{P}(\overline{z}^d) = \left\{p_0 + \overline{z}^d p_1 \colon p_0, p_1 \in \mathcal{P}\right\};$$
$$\mathcal{R}(X, \overline{z}^d) = \left\{g_0 + \overline{z}^d g_1 \colon g_0, g_1 \in \mathcal{R}(X)\right\}.$$

These spaces are modules over the rings $\mathcal{P}$ and $\mathcal{R}(X)$ respectively, generated by the function $\overline{z}^d$. This function is called the generator of $\mathcal{P}(\overline{z}^d)$ and $\mathcal{R}(X, \overline{z}^d)$. For instance, if $d = 1$, then these spaces consist, respectively, of all *bianalytic polynomials* and *bianalytic rational functions* with poles lying outside X (note that bianalytic rational functions are not quotients of bianalytic polynomials).

We are interested in questions about density in the space $C(X)$ of the modules $\mathcal{P}(\overline{z}^d)$ and $\mathcal{R}(Y, \overline{z}^d)$ for some specially chosen compact set $Y \supseteq X$ in the case, when $d > 1$, as well as in questions about density in $C(X)$ of polynomial and rational modules generated by multiple degrees of the function $\overline{z}$:

$$\mathcal{P}(\overline{z}^{k_1}, \dots, \overline{z}^{k_m}) = \left\{p_0 + \overline{z}^{k_1} p_1 + \cdots + \overline{z}^{k_m} p_m \colon p_0, p_1, \dots, p_m \in \mathcal{P}\right\};$$
$$\mathcal{R}(Y, \overline{z}^{k_1}, \dots, \overline{z}^{k_m}) = \left\{g_0 + \overline{z}^{k_1} g_1 + \cdots + \overline{z}^{k_m} g_m \colon g_0, g_1, \dots, g_m \in \mathcal{R}(Y)\right\},$$

where $k_1, \dots, k_m$ are positive integers with $k_1 < \cdots < k_m$.

Of course, if any of the modules that we have defined is dense in the space $C(X)$, then, clearly, the interior of X is empty. Thus, speaking about density of such modules we will always mean their density in the respective appropriately defined subspaces of the space $C(X)$.

The roots of these questions can be traced back to the 1970s–1980s when problems about density of rational modules $\mathcal{R}(Y, \overline{z}, \overline{z}^2, \dots, \overline{z}^n)$ for integer $n \geqslant 1$ were studied by O'Farrell [18], Verdera [22], Carmona [4,5], Wang [24], Trent and Wang [20,21].

Systematical studies of the question about density of polynomial modules $\mathcal{P}(\overline{z}, \overline{z}^2, \dots, \overline{z}^n)$, $n \geqslant 1$, started in the mid-1990s and this question turned out fairly different from the corresponding question about rational approximation, see [15] and references therein. In particular, the approximability conditions in this case cannot be expressed only in terms of topological, metrical or capacitary properties of X. Moreover, several new phenomena related with special analytic properties of planar domains and with special properties of function belonging to model spaces (that is, subspaces of the Hardy space H^2 that are invariant under the backward shift operator) arose in connection with this question.

One ought to notice that in the present paper we allow the situation when the sequence of degrees of generators of our modules has *gaps*. In this situation new interesting connections of problems under consideration with theory of model spaces arise in addition to the initial polyanalytic case.