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On the existence of compacta of minimal capacity in the theory of rational approximation of multi-valued analytic functions

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Abstract

For an interval $E = [a, b]$ on the real line, let μ be either the equilibrium measure, or the normalized Lebesgue measure of E, and let V^{μ} denote the associated logarithmic potential. In the present paper, we construct a function f which is analytic on E and possesses four branch points of second order outside of E such that the family of the admissible compacta of f has no minimizing elements with regard to the extremal theoretic-potential problem, in the external field equals $V^{-\mu}$.

Keywords: Rational approximants; Padé approximation; Orthogonal polynomials; Distribution of poles; Convergence in capacity

1. Notations

Throughout the paper, we use the following notations.

$M(K)$—the space of all positive unit Borel measures μ with supports $S(\mu) = \operatorname{supp}\mu \subset K$, where K is a compact set, $K \subset \overline{\mathbb{C}}$.

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δ_z—the Dirac measure at a point z.

For a finite set $E = \{e_1, \dots, e_n\}$ of points (counted with their multiplicities) on $\overline{\mathbb{C}}$, we introduce the measure $\delta_E := \sum_{k=1}^n \delta_{e_k}$.

For a polynomial Q of degree n, we denote $\delta_Q := \delta_{\{z_1,\dots,z_n\}}$, where $\{z_1, \dots, z_n\}$ are the zeros of Q.

We define a spherically normalized potential $\mathcal{V}^\mu$ of the measure μ by

$$\mathcal{V}^\mu(z) = \int_{|t|\leqslant 1} \log\frac{1}{|z-t|}d\mu(t) + \int_{|t|>1} \log\frac{1}{|1-z/t|}d\mu(t). \tag{1}$$

$\operatorname{cap}_\psi K$ denotes the capacity of the compactum K in the presence of the harmonic external field ψ (the so-called ψ-weighted capacity). It is known [28] that $\operatorname{cap}_\psi K$ coincides with the transfinite diameter $d_\psi K$ of K in the field ψ, that is

$$d_\psi K := \lim_{n\to\infty}\left(\max_{z_1,\dots,z_n\in K}\prod_{1\leqslant q<r\leqslant n}|z_q - z_r|e^{-\left(\psi(z_q)+\psi(z_r)\right)}\right)^{\frac{2}{(n-1)n}}. \tag{2}$$

Let $\psi = \mathcal{V}^{-\mu}$, $S(\mu)\cap K = \varnothing$. For the sake of simplicity we write $\operatorname{cap}_\mu K := \operatorname{cap}_{\mathcal{V}^{-\mu}} K$.

Consider a sequence of compacta $\{K_n\}$, $n = 1, 2, \dots$. It is known [26] that if each K_n is a union of a finite number of continua (the number does not depend on the index n), such that $\lim_{n\to\infty} K_n = K$ (in the Hausdorff metric, see (41)–(42)) and if $S(\mu)\cap K = \varnothing$, then

$$\lim_{n\to\infty} \operatorname{cap}_\mu K_n = \operatorname{cap}_\mu K. \tag{3}$$

$\operatorname{cap} K$ denotes the standard capacity of the compactum $K \subset \mathbb{C}$ in the absence of an external field, i.e. when $\psi \equiv 0$. Since $\mathcal{V}^{\delta_\infty}(z) = 0$ for all $z \in \mathbb{C}$, then $\operatorname{cap} K = \operatorname{cap}_{\delta_\infty} K$.

Given an open set Ω, we say that a sequence of functions $\{R_n\}_{n=1}^\infty$ converges in capacity to the function f on compact subsets of Ω, if for every $K \subset \Omega$ and each $\varepsilon > 0$

$$\lim_{n\to\infty} \operatorname{cap}\{z \in K : \left|(R_n - f)(z)\right| > \varepsilon\} = 0; \tag{4}$$

we use the notation

$$R_n \overset{\text{cap}}{\to} f, \quad z \in \Omega,\ n \to \infty.$$

The notation $\mu_n \overset{*}{\to} \mu$ is used for weak convergence of a sequence of measures μ_n to the measure μ as $n \to \infty$. Recall that

$$\mu_n \overset{*}{\to} \mu \quad \text{as } n \to \infty \iff \lim_{n\to\infty}\int f d\mu_n = \int f d\mu, \tag{5}$$

for each function f continuous on $\overline{\mathbb{C}}$. Also recall that from each sequence of unit measures one can extract a convergent subsequence.

We denote by $g_K(z,\zeta)$ Green's function of the complement $\overline{\mathbb{C}}\setminus K$ of the compactum K, with a pole at $\zeta \in \overline{\mathbb{C}}\setminus K$. (We let $g_K(z,\zeta) = 0$, if the points z and ζ are contained in two separate components of $\overline{\mathbb{C}}\setminus K$.)

$G_K^\mu(z) = \int g_K(z,t)d\mu(t)$ is Green's potential of the measure μ whose support $S(\mu)$ does not intersect the compactum K.

Now let $S(\mu)\cap K = \varnothing$. Let us remind the notion of the balayage $\widetilde{\mu}_K$ of a measure μ onto the compactum K. By definition, the balayage is the unique measure in the space $M(K)$ for which