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Journal of Approximation Theory 205 (2016) 1–24

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Formulas for the Walsh coefficients of smooth functions and their application to bounds on the Walsh coefficients

Kosuke Suzuki*, Takehito Yoshiki

School of Mathematics and Statistics, The University of New South Wales, Sydney, NSW 2052, Australia

Received 18 February 2015; received in revised form 16 November 2015; accepted 25 December 2015
Available online 8 January 2016

Communicated by Josef Dick

Abstract

We establish formulas for the b-adic Walsh coefficients of functions in $C^{\alpha}[0, 1]$ for an integer $\alpha \geq 1$ and give upper bounds on the Walsh coefficients of these functions. We also study the Walsh coefficients of periodic and non-periodic functions in reproducing kernel Hilbert spaces.

Keywords: Walsh series; Walsh coefficient; Sobolev space; Smooth function

1. Introduction

The Walsh coefficients of a function are the generalized Fourier coefficients for the Walsh system, which is an orthonormal system. It is often used instead of the trigonometric Fourier system for analyzing numerical integration [11], approximation [3,10] and constructing low-discrepancy point sets [9,12], especially when we consider as point sets so-called digital nets, which have the suitable group structure for the Walsh system. In particular, the decay of the Walsh coefficients of smooth functions is fundamental to analyzing such problems for spaces of smooth functions. For example, it is used to give explicit constructions of quasi-Monte Carlo rules which achieve

* Corresponding author.
E-mail addresses: kosuke.suzuki1@unsw.edu.au (K. Suzuki), takehito.yoshiki1@unsw.edu.au (T. Yoshiki).

http://dx.doi.org/10.1016/j.jat.2015.12.002
0021-9045/

the optimal rate of convergence for smooth functions in [6,7] and to introduce algorithms to approximate functions in Sobolev spaces [3]. In this paper, we improve previous results from [8] on the decay of the Walsh coefficients of smooth functions.

Throughout the paper we use the following notation: Assume that $b \geq 2$ is a positive integer. We assume that k is a nonnegative integer whose b-adic expansion is

$$k = \kappa_1 b^{a_1 - 1} + \cdots + \kappa_v b^{a_v - 1}$$

where κ_i and a_i are integers with $0 < \kappa_i \leq b - 1$ and $a_1 > \cdots > a_v \geq 1$. For $k = 0$ we assume that $v = 0$ and $a_0 = 0$. We denote by $\mathbb{N}_0$ the set of nonnegative integers. Let $\omega_b := \exp(2\pi\sqrt{-1}/b)$.

The Walsh functions were first introduced by Walsh [17], see also [4,13]. For $k \in \mathbb{N}_0$, the b-adic kth Walsh function $\mathrm{wal}_k(\cdot)$ is defined as

$$\mathrm{wal}_k(x) := \omega_b^{\sum_{i=1}^{v} \kappa_i \xi_{a_i}},$$

for $x \in [0, 1)$ whose b-adic expansion is given by $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$, which is unique in the sense that infinitely many of the digits ξ_i are different from $b - 1$. We also consider s-dimensional Walsh functions. For $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0, 1)^s$, the b-adic $\boldsymbol{k}$th Walsh function $\mathrm{wal}_{\boldsymbol{k}}(\cdot)$ is defined as

$$\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^{s} \mathrm{wal}_{k_j}(x_j).$$

For $\boldsymbol{k} \in \mathbb{N}_0^s$ and $f : [0, 1)^s \to \mathbb{C}$, we define the $\boldsymbol{k}$th Walsh coefficient of f as

$$\widehat{f}(\boldsymbol{k}) := \int_{[0,1)^s} f(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})}\, d\boldsymbol{x}.$$

It is well-known that the Walsh system $\{\mathrm{wal}_{\boldsymbol{k}}(\cdot) \mid \boldsymbol{k} \in \mathbb{N}_0^s\}$ is a complete orthonormal system in $L^2[0, 1)^s$ for any positive integer s (for a proof, see e.g., [11, Theorem A.11]). Hence we have a Walsh series expansion

$$f(\boldsymbol{x}) \sim \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$$

for any $f \in L^2[0, 1)^s$. It is known that if $s = 1$ and $f \in C^0[0, 1]$ has bounded variation then f is equal to its Walsh series expansion, that is, for all $x \in [0, 1)$ we have $f(x) = \sum_{k \in \mathbb{N}_0} \widehat{f}(k) \mathrm{wal}_k(x)$, see [17]. More information on Walsh analysis can be found in the books [14,15]. Hereafter we assume $s = 1$ unless otherwise stated, since the theory of the s-dimensional Walsh coefficients treated in this paper can be reduced to the one-dimensional case by considering coordinate-wise integration (see Theorems 2.5 and 3.9) or tensor product spaces (see Remark 6.6).

There are several studies on the decay of the Walsh coefficients. Fine considered the Walsh coefficients of functions which satisfy a Hölder condition in [13]. Dick studied [6,7] the decay of the Walsh coefficients of functions of smoothness $\alpha \geq 1$ and in more detail in [8]: It was proved that if a function f has $\alpha - 1$ derivatives for which $f^{(\alpha-1)}$ satisfies a Lipschitz condition, then $|\widehat{f}(k)| \leq C b^{-\mu_\alpha(k)}$ for all k, where C is a positive real number which is independent of k and $\mu_\alpha(k) := a_1 + \cdots + a_{\min(\alpha, v)}$. Dick also proved that this order is best possible. That is, for f