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On the zero-free polynomial approximation problem

Arthur A. Danielyan

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA

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Abstract

Let E be a compact set in $\mathbb{C}$ with connected complement, and let $A(E)$ be the class of all complex continuous function on E that are analytic in the interior E^0 of E. Let $f \in A(E)$ be zero free on E^0. By Mergelyan's theorem f can be uniformly approximated on E by polynomials, but is it possible to realize such approximation by polynomials that are zero-free on E? This natural question has been proposed by J. Andersson and P. Gauthier. So far it has been settled for some particular sets E. The present paper describes classes of functions for which zero free approximation is possible on an arbitrary E.

1. Introduction and the main result

For a set M in the complex plane $\mathbb{C}$ we denote as usual by M^0, ∂M, and $\overline{M}$ the interior, the boundary, and the closure of M, respectively.

Suppose E is an arbitrary compact subset in $\mathbb{C}$ such that $\mathbb{C} \setminus E$ is connected. Let $A(E)$ be the usual space (algebra) of all complex-valued continuous functions on E that are analytic in E^0. The following well-known approximation theorem is due to S.N. Mergelyan (see [10] or [12]).

Theorem A. *Let $f \in A(E)$. Then for each $\epsilon > 0$ there exists a polynomial $P(z)$ such that $|f(z) - P(z)| < \epsilon$ for all $z \in E$.*

In regard to this theorem, the following natural question is on the possibility of approximation by polynomials that are zero-free on the set of approximation.

E-mail address: adaniely@usf.edu.

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Question 1. Let $f \in A(E)$ have no zeros on E^0 and let $\epsilon > 0$. Does there exist a polynomial $P(z)$ with no zeros on E such that $|f(z) - P(z)| < \epsilon$ for all $z \in E$?

Note that if $f \in A(E)$ has an isolated zero at a point of E^0, then by Hurwitz's theorem for such f a zero free polynomial approximation is impossible.

Question 1 has been proposed by J. Andersson and P. M. Gauthier (cf. [1]) and it has been investigated in the recent papers [2,1,3,6–9]. Several interesting results of affirmative character have been proved under various restrictions on E, but the question still remains open in the general case. Without going into details let us mention that P. Gauthier and G. Knese [8] have affirmatively settled Question 1 in the case when E is a chain of (closed) Jordan domains. Another progress in this direction is due to S. Khrushchev [9], who has extended the result of [8] for the more general class of locally connected compact sets E with connected complement.

The present paper describes classes of functions defined on an arbitrary compact set E with connected complement for which zero free approximation is possible.

To avoid any possible confusion, let us first recall (introduce) some terminology.

Definition 1. Let $E \subset \mathbb{C}$ be a compact set. We call $H \subset \partial E$ a zero set if some $g \in A(E)$ vanishes precisely on H (that is, $H = \{z \in E : g(z) = 0\}$).

Clearly any such H is a closed subset on ∂E and also the function g is zero free on E^0.

The main result of this paper is the following:

Theorem 1. *Let $E \subset \mathbb{C}$ be a compact set with connected complement and let $H \subset \partial E$ be a zero set. Then there exists $f \in A(E)$ with H as its zero set and allowing uniform approximation by polynomials zero free on E.*

The theorem implies that for any prescribed zero set H the zero free approximation is possible at least for some functions. In particular, a possible counterexample to Question 1, has to depend, besides the set H, also on a specific function of $A(E)$ vanishing on H. (Since in the general case both E and H may be complicated sets, the last conclusion sheds some light on the nature of a possible counterexample; see also the discussion in [1, Sect. 4].)

Each polynomial $P(z) = a(z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_n)$ can be considered as a continuous function of its zeros $\zeta_1, \zeta_2, \ldots, \zeta_n$ (and z). This, combined with the fact that ∂E is nowhere dense in $\mathbb{C}$, clearly implies that Question 1 is equivalent to the following question.

Question 1′. Let $f \in A(E)$ have no zeros on E^0 and let $\epsilon > 0$. Does there exist a polynomial $P(z)$ with no zeros on E^0 such that $|f(z) - P(z)| < \epsilon$ if $z \in E$?

Assume $E^0 = \emptyset$ and $f \in A(E)$. Then by Theorem A (or merely by its particular case known as Lavrentiev's theorem) a polynomial approximation to f is possible, and trivially Question 1′ has an affirmative answer (since $E^0 = \emptyset$). Then the same is true for Question 1 and thus we have the following result of [2] (cf. also [1]):

Proposition 1. *If $E^0 = \emptyset$, zero free polynomial approximation on E is always possible.*

If $E^0 \neq \emptyset$ but $f \in A(E)$ is zero free on $\overline{E^0}$ then obviously by Theorem A one can approximate f on E by a polynomial which is zero free on $\overline{E^0}$. This implies, in particular, that Question 1′ has an affirmative answer. Then, as above, Question 1 too has an affirmative answer and we arrive at the following extension of Proposition 1: