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Hankel determinants for a singular complex weight and the first and third Painlevé transcendents

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Abstract

In this paper, we consider polynomials orthogonal with respect to a varying perturbed Laguerre weight $e^{-n(z-\log z+t/z)}$ for $t < 0$ and z on certain contours in the complex plane. When the parameters n, t and the degree k are fixed, the Hankel determinant for the singular complex weight is shown to be the isomonodromy τ-function of the Painlevé III equation. When the degree $k = n$, n is large and t is close to a critical value, inspired by the study of the Wigner time delay in quantum transport, we show that the double scaling asymptotic behaviors of the recurrence coefficients and the Hankel determinant are described in terms of a Boutroux tronquée solution to the Painlevé I equation. Our approach is based on the Deift–Zhou nonlinear steepest descent method for Riemann–Hilbert problems.

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1. Introduction and statement of results

Let $w_t(x)$ be the following singularly perturbed Laguerre weight

$$w_t(x) = w(x;t) = e^{-nV_t(x)}, \quad x \in (0, +\infty) \tag{1.1}$$

with

$$V_t(x) = x - \log x + \frac{t}{x}, \quad t \geq 0. \tag{1.2}$$

The Hankel determinant is defined as

$$D_k[w(x;t)] = \det(\mu_{i+j})_{i,j=0}^{k-1}, \tag{1.3}$$

where μ_j is the jth moment of $w_t(x)$, namely,

$$\mu_j = \int_0^\infty x^j w_t(x) dx.$$

Note that when $t \geq 0$, the integral in the above formula is convergent so that the Hankel determinant $D_k[w;t] = D_k[w(x;t)]$ in (1.3) is well-defined. Moreover, it is well-known that the Hankel determinant can be expressed as

$$D_k[w;t] = \prod_{j=0}^{k-1} \gamma_{j,n}^{-2}(t); \tag{1.4}$$

see [26, p. 28], where $\gamma_{k,n}(t)$ is the leading coefficient of the kth order polynomial orthonormal with respect to the weight function in (1.1). Or, let $\pi_{k,n}(x)$ be the kth order monic orthogonal polynomial, then $\gamma_{k,n}(t)$ appears in the following orthogonal relation

$$\int_0^\infty \pi_{k,n}(x) x^j e^{-nV_t(x)} dx = \gamma_{k,n}^{-2}(t)\delta_{jk}, \quad j = 0, 1, \ldots, k$$

for fixed n. Moreover, the monic orthogonal polynomials $\pi_{k,n}(x)$ satisfy a three-term recurrence relation as follows:

$$x\pi_{k,n}(x) = \pi_{k+1,n}(x) + \alpha_{k,n}(t)\pi_{k,n}(x) + \beta_{k,n}(t)\pi_{k-1,n}(x), \quad k = 0, 1, \ldots, \tag{1.5}$$

with $\pi_{-1,n}(x) \equiv 0$ and $\pi_{0,n}(x) \equiv 1$, where the appearance of n and t in the coefficients indicates their dependence on n and the parameter t in the varying weight (1.1).

In this paper, however, we will focus on the case when $t < 0$. Since all the above integrals on $[0, \infty)$ become divergent for negative t, we need to deform the integration path from the positive real axis to certain curves in the complex plane. Consequently, the orthogonality will be converted to the *non-Hermitian orthogonality* in the complex plane. More precisely, let us define the following new weight function on $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$:

$$w_t(z) = w(z;t) = c_j e^{-nV_t(z)}, \quad z \in \Gamma_j, \text{ with } c_1 = 1, c_2 = \alpha, c_3 = 1 - \alpha, \tag{1.6}$$

where α is a complex constant, the curves $\Gamma_1 = (2\delta, \infty)$, $\Gamma_2 = \{\delta(1 + e^{i\theta}) | \ \theta \in (0, \pi)\}$ and $\Gamma_3 = \{\delta(1 + e^{i\theta}) | \ \theta \in (-\pi, 0)\}$; see Fig. 1, δ being a positive constant. The potential is defined in the cut plane $\mathbb{C} \setminus (-\infty, 0]$ as

$$V_t(z) = z - \log z + \frac{t}{z}, \quad \arg z \in (-\pi, \pi), \ t < 0. \tag{1.7}$$