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On orthonormal bases and translates

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Abstract

We construct an orthonormal basis in $L_2(\mathbb{R})$ by integer translations of elements of a convergent sequence of functions.

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1. Introduction

It is well-known [4] that a system of translates of a single function cannot be an orthonormal basis, nor a Riesz one, in the space $L_2(\mathbb{R})$. Moreover [2], shifts of finitely many functions never generate even a frame.

H. Shapiro posed a question [5]: does there exist an orthonormal basis obtained by translations from a compact set of functions? In this note we give a positive answer to this question, in a slightly stronger form:

Theorem. *There exists a set of functions* $\Phi = \{\phi_n(t)\}$, $n \in \mathbb{N}$, *in the space* $L_2(\mathbb{R})$ *such that* $\|\phi_n - \mathbf{1}_{[0,1]}\| \to 0$, *and* $\{\phi_n(t-n)\}$ *is an orthonormal basis.*

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To prove it, we introduce the following $(n+1)\times(n+1)$ matrix:

$$A_n = \begin{pmatrix} 1-\gamma_n & -\gamma_n & \cdots & -\gamma_n & \dfrac{1}{\sqrt{2n}} \\ -\gamma_n & 1-\gamma_n & \cdots & -\gamma_n & \dfrac{1}{\sqrt{2n}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\gamma_n & -\gamma_n & \cdots & 1-\gamma_n & \dfrac{1}{\sqrt{2n}} \\ -\dfrac{1}{\sqrt{2n}} & -\dfrac{1}{\sqrt{2n}} & \cdots & -\dfrac{1}{\sqrt{2n}} & \dfrac{1}{\sqrt{2}} \end{pmatrix},$$

where $\gamma_n = \frac{1}{(2+\sqrt{2})\,n}$. It is easy to check that the matrix is orthogonal.

The main point here is that the upper left $n \times n$ submatrix of A_n is close to the identity matrix while the lower right element is essentially less than 1.

2. Proof of theorem

Our construction below is inspired by Bourgain's paper [1]. We will use the following

Lemma. *Let $\Psi = \{\psi_n\}$ be a set in a Hilbert space. For some $\alpha < 1$, suppose there is a set Γ, dense in the unit sphere, such that every $g \in \Gamma$ can be approximated, with an error less than α, by a linear combination of vectors ψ_n. Then Ψ is complete.*

Proof. If Ψ were not complete, then there would be a vector f, $\|f\| = 1$, orthogonal to $\operatorname{span}(\Psi)$. Take $g \in \Gamma$ with $\|f - g\| < 1 - \alpha$, and find $\psi \in \operatorname{span}(\Psi)$ with $\|g - \psi\| < \alpha$. Then $\|f - \psi\| < 1$ which contradicts the choice of f. •

Let $\Gamma = \{g_k\}$, $k \in \mathbb{N}$, be a sequence of functions in $L_2(\mathbb{R})$ dense in the unit sphere S, with two additional properties: $g_k = 0$ a.e. outside $[-k, k]$ and

$$\|g_k\, \mathbf{1}_{[-k,-k+1]}\| > 0.$$

It can be made, e.g., by appropriate rearrangement of a countable dense in S set of compactly supported functions, perturbing a kth one by $\pm\frac{1}{k}\mathbf{1}_{[-k,-k+1]}$, and normalizing.

The desired orthonormal basis will be built up inductively. Fix a sequence of integers $0 < n_1 < n_2 < \cdots$.

Step 1. Take $n = n_1$, and apply the matrix A_n to the orthonormal set of functions $\chi_1^{(1)} = \mathbf{1}_{[1,2]}$, $\chi_2^{(1)} = \mathbf{1}_{[2,3]}, \ldots, \chi_n^{(1)} = \mathbf{1}_{[n,n+1]}$, and $g^{(1)} = g_1$:

$$A_n \begin{pmatrix} \chi_1^{(1)} \\ \vdots \\ \chi_n^{(1)} \\ g^{(1)} \end{pmatrix} = \begin{pmatrix} \psi_1^{(1)} \\ \vdots \\ \psi_n^{(1)} \\ h^{(1)} \end{pmatrix}.$$