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The spectral analysis of three families of exceptional Laguerre polynomials

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Abstract

The Bochner Classification Theorem (1929) characterizes the polynomial sequences $\{p_n\}_{n=0}^{\infty}$, with deg $p_n = n$ that simultaneously form a complete set of eigenstates for a second-order differential operator and are orthogonal with respect to a positive Borel measure having finite moments of all orders. Indeed, up to a complex linear change of variable, only the classical Hermite, Laguerre, and Jacobi polynomials, with certain restrictions on the polynomial parameters, satisfy these conditions. In 2009, Gómez-Ullate, Kamran, and Milson found that for sequences $\{p_n\}_{n=1}^{\infty}$, deg $p_n = n$ (without the constant polynomial), the only such sequences satisfying these conditions are the *exceptional* X_1-Laguerre and X_1-Jacobi polynomials. Subsequently, during the past five years, several mathematicians and physicists have discovered and studied other exceptional orthogonal polynomials $\{p_n\}_{n\in\mathbb{N}_0\setminus A}$, where A is a finite subset of the non-negative integers $\mathbb{N}_0$ and where deg $p_n = n$ for all $n \in \mathbb{N}_0\setminus A$. We call such a sequence an exceptional polynomial sequence of codimension $|A|$, where the latter denotes the cardinality of A. All exceptional sequences with a non singular weight, found to date, have the remarkable feature that they form a complete orthogonal set in their natural Hilbert space setting.

Among the exceptional sets already known are two types of exceptional Laguerre polynomials, called the Type I and Type II exceptional Laguerre polynomials, each omitting m polynomials. In this paper, we briefly discuss these polynomials and construct the self-adjoint operators generated by their corresponding

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second-order differential expressions in the appropriate Hilbert spaces. In addition, we present a novel derivation of the Type III family of exceptional Laguerre polynomials along with a detailed disquisition of its properties. We include several representations of these polynomials, orthogonality, norms, completeness, the location of their local extrema and roots, root asymptotics, as well as a complete spectral study of the second-order Type III exceptional Laguerre differential expression.

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1. Introduction

An *exceptional orthogonal polynomial system* is a sequence $\{p_n\}_{n\in\mathbb{N}_0\setminus A}$ with the following characteristic properties:

(a) $\deg(p_n) = n$ for $n \in \mathbb{N}_0 \setminus A$, where A is a finite subset of $\mathbb{N}_0$;
(b) there exists an interval $I = (a, b)$ and a Lebesgue measurable weight $w > 0$ on I such that

$$\int_I p_n p_m w = k_n \delta_{n,m} \quad (n, m \in \mathbb{N}_0 \setminus A)$$

for some $k_n > 0$; here $\delta_{n,m}$ denotes the Kronecker delta symbol;
(c) there exists a second-order differential expression

$$\ell[y](x) = a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x)$$

and, for each $n \in \mathbb{N}_0 \setminus A$, there exists a $\lambda_n \in \mathbb{C}$ such that $y = p_n(x)$ is a solution of

$$\ell[y](x) = \lambda_n y(x) \quad (x \in I);$$

(d) for $n \in A$, there does not exist a polynomial p of degree n such that $y = p(x)$ satisfies $\ell[y] = \lambda y$ for any choice of $\lambda \in \mathbb{C}$;
(e) all of the moments

$$\int_I x^n w(x)dx \quad (n \in \mathbb{N}_0)$$

of w exist and are finite.

If $|A|$ denotes the cardinality of the set A, we call a sequence $\{p_n\}_{n\in\mathbb{N}_0\setminus A}$ satisfying conditions (a)–(e) above an *exceptional polynomial sequence of codimension* $|A|$. The case $A = \{0\}$ was treated in [12], where the authors classified exceptional orthogonal polynomials with one missing degree, and introduced the X_1-Laguerre and the X_1-Jacobi polynomials, so named because of their similarity to their classical cousins. The fact that these sequences omit a constant polynomial distinguishes their characterization from the Bochner classification [2] characterizing the Jacobi, Laguerre, and Hermite polynomials; of course, the Bochner classification corresponds to $A = \varnothing$. Since 2009, several authors have generalized the results of Kamran, Milson and Gómez-Ullate in [12] by finding other sequences of exceptional polynomials $\{p_n\}_{n\in\mathbb{N}_0\setminus A}$, where A is a finite subset of $\mathbb{N}_0$, satisfying each of the conditions in (a)–(e).

In this paper, we discuss three families of exceptional Laguerre polynomials, each spanning a flag of codimension m. Specifically, we deal with two such exceptional Laguerre sequences