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Characterization of 1-quasi-greedy bases

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Abstract

In this note we continue the study initiated in Albiac and Wojtaszczyk (2006) of greedy-like bases in the "isometric case", i.e., in the case that the constants that arise in the context of greedy bases in their different forms are 1. Here we settle the problem to find a satisfactory characterization of 1-quasi-greedy bases in Banach spaces. We show that a semi-normalized basis in a Banach space is quasi-greedy with quasi-greedy constant 1 if and only if it is unconditional with suppression-unconditional constant 1.

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1. Introduction and background

Let $(\mathbb{X}, \|\cdot\|)$ be an infinite-dimensional (real or complex) Banach space, and let $\mathcal{B} = (\mathbf{e}_n)_{n=1}^\infty$ be a semi-normalized basis for $\mathbb{X}$ with biorthogonal functionals $(\mathbf{e}_n^*)_{n=1}^\infty$. The basis $\mathcal{B}$ is *quasi-greedy* (see [10]) if for any $x \in \mathbb{X}$ the corresponding series expansion,

$$x = \sum_{n=1}^{\infty} \mathbf{e}_n^*(x)\mathbf{e}_n$$

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converges in norm after reordering it so that the sequence $(|\mathbf{e}_n^*(x)|)_{n=1}^\infty$ is decreasing. Wojtaszczyk showed [11] that a basis $(\mathbf{e}_n)_{n=1}^\infty$ of $\mathbb{X}$ is quasi-greedy if and only if the greedy operators $\mathcal{G}_N\colon \mathbb{X} \to \mathbb{X}$ defined by

$$x = \sum_{j=1}^{\infty} \mathbf{e}_j^*(x)\mathbf{e}_j \mapsto \mathcal{G}_N(x) = \sum_{j\in\Lambda_N(x)} \mathbf{e}_j^*(x)\mathbf{e}_j,$$

where $\Lambda_N(x)$ is any N-element set of indices such that

$$\min\{|\mathbf{e}_j^*(x)|\colon j \in \Lambda_N(x)\} \ge \max\{|\mathbf{e}_j^*(x)|\colon j \notin \Lambda_N(x)\},$$

are uniformly bounded, i.e.,

$$\|\mathcal{G}_N(x)\| \le C\|x\|, \quad x \in \mathbb{X},\ N \in \mathbb{N}, \tag{1}$$

for some constant C independent of x and N. Note that the operators $(\mathcal{G}_N)_{N=1}^\infty$ are neither linear nor continuous, so this is not just the Uniform Boundedness Principle!

Obviously, (1) implies that then there is a (possibly different) constant $\tilde{C}$ such that

$$\|x - \mathcal{G}_N(x)\| \le \tilde{C}\|x\|, \quad x \in \mathbb{X},\ N \in \mathbb{N}. \tag{2}$$

We will denote by $C_w = C_w[\mathcal{B}, \mathbb{X}]$ the smallest constant such that (1) holds, and by $C_\ell = C_\ell[\mathcal{B}, \mathbb{X}]$ the least constant in (2). We will refer to C_ℓ as the *suppression quasi-greedy constant of the basis*. It is rather common (cf. [7,3]) and convenient to define the *quasi-greedy constant* of the basis as

$$C_{qg} = C_{qg}[\mathcal{B}, \mathbb{X}] = \max\{C_w[\mathcal{B}, \mathbb{X}],\ C_\ell[\mathcal{B}, \mathbb{X}]\}.$$

If $\mathcal{B}$ is a quasi-greedy basis and C is a constant such that $C_{qg} \le C$ we will say that $\mathcal{B}$ is *C-quasi-greedy*.

Recall also that a basis $(\mathbf{e}_n)_{n=1}^\infty$ in a Banach space $\mathbb{X}$ is *unconditional* if for any $x \in \mathbb{X}$ the series $\sum_{n=1}^\infty \mathbf{e}_n^*(x)\mathbf{e}_n$ converges in norm to x regardless of the order in which we arrange the terms. The property of being unconditional is easily seen to be equivalent to that of being *suppression unconditional*, which means that the natural projections onto any subsequence of the basis

$$P_A(x) = \sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n, \quad A \subset \mathbb{N},$$

are uniformly bounded, i.e., there is a constant K such that for all $x = \sum_{n=1}^\infty \mathbf{e}_n^*(x)\mathbf{e}_n$ and all $A \subset \mathbb{N}$,

$$\left\|\sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n\right\| \le K \left\|\sum_{n=1}^{\infty} \mathbf{e}_n^*(x)\mathbf{e}_n\right\|. \tag{3}$$

The smallest K in (3) is the *suppression unconditional constant* of the basis, and will be denoted by $K_{su} = K_{su}[\mathcal{B}, \mathbb{X}]$. Notice that

$$K_{su}[\mathcal{B}, \mathbb{X}] = \sup\{\|P_A\|\colon A \subset \mathbb{N} \text{ is finite}\} = \sup\{\|P_A\|\colon A \subset \mathbb{N} \text{ is cofinite}\}.$$

If a basis $\mathcal{B}$ is unconditional and K is a constant such that $K_{su} \le K$ we will say that $\mathcal{B}$ is *K-suppression unconditional*.