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Optimal algorithms for doubly weighted approximation of univariate functions

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Abstract

We consider a ϱ-weighted L_q approximation in the space of univariate functions $f : \mathbb{R}_+ \to \mathbb{R}$ with finite $\|f^{(r)}\psi\|_{L_p}$. Let $\alpha = r - 1/p + 1/q$ and $\omega = \varrho/\psi$. Assuming that ψ and ω are non-increasing and the quasi-norm $\|\omega\|_{L_{1/\alpha}}$ is finite, we construct algorithms using function/derivatives evaluations at n points with the worst case errors proportional to $\|\omega\|_{L_{1/\alpha}} n^{-r+(1/p-1/q)_+}$. In addition we show that this bound is sharp; in particular, if $\|\omega\|_{L_{1/\alpha}} = \infty$ then the rate $n^{-r+(1/p-1/q)_+}$ cannot be achieved. Our results generalize known results for bounded domains such as $[0, 1]$ and $\varrho = \psi \equiv 1$. We also provide a numerical illustration.

Keywords: Function approximation; Unbounded domains; Optimal algorithms

1. Introduction

We study in this paper the approximation of univariate real-valued functions $f : D \to \mathbb{R}$ where the domain is $D = \mathbb{R}_+ = [0, \infty)$ and the error of approximation is measured in a

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ϱ-weighted L_q (semi-)norm,

$$\|f\varrho\|_{L_q} = \left(\int_D |f(x)\,\varrho(x)|^q\,\mathrm{d}x\right)^{1/q}, \quad 1 \le q \le \infty.$$

Here $\varrho : D \to \mathbb{R}_+$ is a nonnegative and measurable weight function. The restriction to $D = [0, \infty)$ is to simplify the notation only, since all the results can be easily extended to D being an arbitrary interval including $D = \mathbb{R}$.

We assume that approximation algorithms use function and/or derivatives values at n points, and we study the worst case errors of such algorithms with respect to the unit balls of the following spaces $F = F(r, p, \psi)$. For given positive integer r, $1 \le p \le \infty$, and a positive and measurable weight function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, the space F consists of functions with (locally) absolutely continuous derivative $f^{(r-1)}$ and

$$\|f^{(r)}\psi\|_{L_p} < \infty.$$

A special case of such a problem is the unweighted approximation on a compact interval $[0, T]$, which corresponds to $\psi \equiv 1$ and $\varrho(x) = 1$ for $x \in [0, T]$ and $\varrho(x) = 0$ otherwise. It follows from, e.g., [2,8,13,14] that the nth minimal worst case errors are then proportional to

$$T^{\,r-1/p+1/q}\, n^{-r+(1/p-1/q)_+} \quad \text{where } x_+ = \max(0, x). \tag{1}$$

Note that for $p = r = 1$ and $q = \infty$ the errors do not converge to zero; therefore this case is excluded from our considerations.

Doubly weighted approximation problems were first investigated in [16], see also [9–12]. Moreover, the spaces $F(r, p, \psi)$ and the corresponding results were used to construct weighted tensor product spaces of multivariate and ∞-variate functions, see, e.g., [4,3,5,6,15,17].

The results of [16] were obtained under rather complicated assumptions. In the current paper, we obtain more accurate results using simpler assumptions and deliver different algorithms from those in [16]. Assumptions and results of this paper are rather comparable to those in [11], where the weighted integration problem with $r = 1$, $p = \infty$, $\psi \equiv 1$, and the weight $\varrho(x) = \exp(-x)$ was considered; see also [1,10,12]. We believe that our new algorithms are more suitable for constructing Smolyak's (often called Sparse Grid) algorithms for multivariate approximation problems.

We now discuss the main results of the paper. Define

$$\alpha = r - \frac{1}{p} + \frac{1}{q} \quad \text{and} \quad \omega = \frac{\varrho}{\psi}.$$

We assume that ψ and ω are monotonically non-increasing, and that

$$\|\omega^{1/\alpha}\|_{L_1} = \int_{\mathbb{R}_+} \omega^{1/\alpha}(x)\,\mathrm{d}x < \infty.$$

Clearly, $\|\omega^{1/\alpha}\|_{L_1}^{\alpha} = \|\omega\|_{L_{1/\alpha}}$ is the $L_{1/\alpha}$ *quasi*-norm of ω.

For given $n \ge 1$, let the points $x_{n,i}$ for $i = 1, \ldots, n$ be given by

$$\int_0^{x_{n,i}} \omega^{1/\alpha}(x)\,\mathrm{d}x = \frac{i-1}{n}\,\|\omega^{1/\alpha}\|_{L_1}.$$

That is, $x_{n,i}$ are chosen so that the integrals of $\omega^{1/\alpha}$ between successive points are equal. We prove that the algorithm based on piecewise Taylor polynomials of degree $r-1$ at the points $x_{n,i}$