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Journal of Approximation Theory 200 (2015) 28–39

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Zeros of exceptional Hermite polynomials

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Received 16 December 2014; received in revised form 19 June 2015; accepted 10 July 2015
Available online 20 July 2015

Communicated by Francisco Marcellan

Abstract

We study the zeros of exceptional Hermite polynomials associated with an even partition λ. We prove several conjectures regarding the asymptotic behaviour of both the regular (real) and the exceptional (complex) zeros. The real zeros are distributed as the zeros of usual Hermite polynomials and, after contracting by a factor $\sqrt{2n}$, we prove that they follow the semi-circle law. The non-real zeros tend to the zeros of the generalized Hermite polynomial H_λ, provided that these zeros are simple. It was conjectured by Veselov that the zeros of generalized Hermite polynomials are always simple, except possibly for the zero at the origin, but this conjecture remains open.

Keywords: Exceptional orthogonal polynomials; Hermite polynomials; Zero distribution

1. Introduction

The field of classical orthogonal polynomials is essentially the study of Sturm–Liouville problems with polynomial solutions. Indeed, by the well-known theorem of Bochner, if we assume that a Sturm–Liouville problem admits an eigenpolynomial of *every* degree, then we arrive at the well-known families of Hermite, Laguerre, Jacobi, and Bessel (if signed weights are allowed). Exceptional orthogonal polynomials arise when we consider Sturm–Liouville problems

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http://dx.doi.org/10.1016/j.jat.2015.07.002

with polynomial eigenfunctions, but allow a finite number of degrees to be missing from the corresponding degree sequence [12]. For background on exceptional orthogonal polynomials and exact solutions in quantum mechanics, see [13,21]. For recent developments in the area of exceptional Hermite polynomials, see [7,11].

The study of the zeros of exceptional orthogonal polynomials has attracted some recent interest. Preliminary results indicate that there are strong parallels with the behaviour of zeros of classical orthogonal polynomials. For example, it is possible to describe the zeros of certain exceptional OP using an electrostatic interpretation [5,16]. Asymptotic behaviour of the zeros of 1-step Laguerre and Jacobi exceptional polynomials as the degree n goes to infinity was considered in [14,15,18].

All known families of exceptional OP have a weight of the form

$$W(x) = \frac{W_0(x)}{\eta(x)^2} \tag{1.1}$$

where $W_0(x)$ is a classical OP weight and where $\eta(x)$ is a certain polynomial whose degree is equal to the number of gaps in the XOP degree sequence, and which does not vanish on the domain of orthogonality. It has recently been shown [10] that the weight indeed takes the above form for every exceptional OP family.

The zeros of exceptional orthogonal polynomials are divided into two groups according to whether they lie in the domain of orthogonality. The *regular* zeros are of this type and enjoy the usual intertwining behaviour common to solutions of all Sturm–Liouville problems. All other types of zeros are called *exceptional zeros*. For sufficiently high degree n, the number of exceptional zeros is precisely equal to the degree of $\eta(x)$. Based on all extant investigations of the asymptotics of the zeros of exceptional OP it is reasonable to formulate the following.

Conjecture 1.1. *The regular zeros of exceptional OP have the same asymptotic behaviour as the zeros of their classical counterpart. The exceptional zeros converge to the zeros of the denominator polynomial* $\eta(x)$.

The first part of this conjecture admits two useful interpretations. The simplest interpretation is that after suitable normalization, the kth regular zero of the exceptional polynomials converges to the kth zero of their classical counterpart. Such asymptotic behaviour can be proved by means of Mehler–Heine type theorems, as was done for the case of certain exceptional Laguerre and Jacobi polynomials in [14,18].

However, there is another way to formulate this conjecture. It is well known that as the degree goes to infinity, the counting measure for the zeros of classical orthogonal polynomials, suitably normalized, tends to a certain equilibrium measure. The conjecture then is that the normalized counting measure of the regular zeros of exceptional orthogonal polynomials converges to the same equilibrium measure.

Exceptional Hermite polynomials are Wronskians of $r + 1$ classical Hermite polynomials, where r of the polynomials are fixed and where the degree of the last polynomial varies. In the Hermite case, the denominator $\eta(x)$ in (1.1) is just the Wronskian of the r fixed polynomials. The equilibrium measure for Hermite polynomials is known as the semicircle law. The relevance of this measure was already noted in [8], which considered asymptotics of certain second- and third-order Wronskians of Hermite polynomials. That paper is also notable because it formulates an important conjecture, stated precisely in our paper as Conjecture 2.4, about the simplicity of zeros of such Wronskians.