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ScienceDirect
Journal of Approximation Theory 200 (2015) 92–135
Journal of Approximation Theory
www.elsevier.com/locate/jat

Full length article

Strong estimates of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated Boolean sums

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Received 11 February 2015; received in revised form 22 July 2015; accepted 28 July 2015
Available online 4 August 2015

Communicated by Dany Leviatan

Abstract

We establish matching direct and two-term strong converse estimates of the rate of weighted simultaneous approximation by the Bernstein operator and its iterated Boolean sums for smooth functions in L_p-norm, $1 < p \le \infty$. We consider Jacobi weights. The characterization is stated in terms of appropriate moduli of smoothness or K-functionals. Also, analogous results concerning the generalized Kantorovich operators are derived.

Keywords: Bernstein polynomial; Kantorovich polynomial; Boolean sum; Simultaneous approximation; Rate of convergence

1. Main results

The Bernstein operator is defined for $f \in C[0,1]$ and $x \in [0,1]$ by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \qquad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

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http://dx.doi.org/10.1016/j.jat.2015.07.006

As is known its saturation order is n^{-1} and the differential operator which describes its rate of approximation is $Df(x) = \varphi^2(x)f''(x)$ with $\varphi(x) = \sqrt{x(1-x)}$ (see e.g. [4, Chapter 10, Theorems 3.1 and 5.1]). More precisely, Voronovskaya's classic result states

$$\lim_{n\to\infty} n\,(B_n f(x) - f(x)) = \frac{1}{2} Df(x) \quad \text{uniformly on } [0,1] \tag{1.1}$$

for $f \in C^2[0,1]$.

One way to increase the approximation rate of B_n is to form its iterated Boolean sums $\mathcal{B}_{r,n} : C[0,1] \to C[0,1]$, defined by

$$\mathcal{B}_{r,n} = I - (I - B_n)^r,$$

where I stands for the identity and $r \in \mathbb{N}$. In [28] it was shown that their saturation order is n^{-r}. Gonska and Zhou [18] established a neat direct estimate for $\mathcal{B}_{r,n}$ and a Stechkin-type inverse inequality. Also, they made a historical overview of the study of that kind of operators and explained why they can be regarded as iterated Boolean sums. Later on Ding and Cao [5] characterized the error of the multivariate generalization of $\mathcal{B}_{r,n}$ on the simplex and further improved the lower estimate. They used a K-functional with a differential operator that reduces in the univariate case to D^r, i.e. exactly the rth iterate of the differential operator associated with B_n as it should be expected.

Here we shall consider simultaneous approximation by $\mathcal{B}_{r,n}$. It is known that the derivatives of the Bernstein polynomial of a smooth function approximate the corresponding derivatives of the function (see [4, Chapter 10, Theorem 2.1]). López-Moreno, Martínez-Moreno and Muñoz-Delgado [24] and Floater [14] extended (1.1) showing that for $f \in C^{s+2}[0,1]$ we have

$$\lim_{n\to\infty} n\left((B_n f(x))^{(s)} - f^{(s)}(x)\right) = \frac{1}{2}(Df(x))^{(s)} \quad \text{uniformly on } [0,1]. \tag{1.2}$$

Hence the differential operator that describes the simultaneous approximation by B_n is $(d/dx)^s D$. Results about the rate of convergence in (1.2) were established in [15–17].

So, it is reasonable to expect that the differential operator related to the simultaneous approximation by $\mathcal{B}_{r,n}$ is $(d/dx)^s D^r$ and the saturation order is n^{-r}. That turns out to be indeed so. Before stating our main results let us note that, since the derivative of the Bernstein polynomial is closely related to the Kantorovich polynomial, it makes sense to consider approximation not only in the uniform norm but also in the L_p-norm. Moreover, weights of the form $\varphi^{2\ell}$ with $\ell \in \mathbb{N}$ appear naturally in the study of the approximation rate of $\mathcal{B}_{r,n}$ (see the proof of [18, Theorem 1(i)] we gave in [9, pp. 35–36]). So, it is appropriate to consider simultaneous approximation by $\mathcal{B}_{r,n}$ with Jacobi weights. We set

$$w(x) = w(\gamma_0, \gamma_1; x) = x^{\gamma_0}(1-x)^{\gamma_1}, \quad x \in (0,1), \tag{1.3}$$

where $\gamma_0, \gamma_1 > -1/p$ for $1 \le p < \infty$ or $\gamma_0, \gamma_1 \ge 0$ for $p = \infty$. To characterize the rate of the simultaneous approximation by $\mathcal{B}_{r,n}$, we shall use the K-functional

$$K_{r,s}(f,t)_{w,p} = \inf_{g \in C^{2r+s}[0,1]} \left\{ \|w(f - g^{(s)})\|_p + t\|w(D^r g)^{(s)}\|_p \right\}.$$

We denote by $\|f\|_p$ the L_p-norm of the function f on the interval $[0,1]$. When the norm is taken on a subinterval $J \subset [0,1]$, we shall write $\|f\|_{p(J)}$. As usual, $AC^k[a,b]$ stands for the set of all functions, which along with their derivatives up to order $k \in \mathbb{N}_0$ are absolutely continuous