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Journal of Approximation Theory 200 (2015) 221–226

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Finite Hilbert transforms

Dang Vu Giang

Hanoi Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, 10307 Hanoi, Viet Nam

Received 12 September 2013; received in revised form 14 July 2015; accepted 10 August 2015
Available online 18 August 2015

Communicated by Vilmos Totik

Abstract

Several interesting formulas concerning finite Hilbert transform and logarithmic integrals are proved with application determining equilibrium measures.

MSC: primary 42A20-38; secondary 44A15

Keywords: Hilbert transform; Complex Hardy spaces; Boundary functions; Equilibrium measures

1. Complex Hardy spaces and boundary functions

Recall that the Hilbert transform $Hf = \tilde{f}$ of a function $f \in L^p(\mathbb{R})$ $(1 \le p < \infty)$ is defined by letting

$$Hf(x) = \tilde{f}(x) = \frac{1}{\pi}\,(\text{p.v.}) \int_{-\infty}^{\infty} \frac{f(t)}{x-t} \cdot dt.$$

We use both notations Hf and $\tilde{f}$ for the Hilbert transform of a function f. For example, the Hilbert transform of the characteristic function $\chi_{(a,b)}$ of the interval (a, b) is

$$\tilde{\chi}_{(a,b)}(x) = \frac{1}{\pi} \cdot \ln\left|\frac{x-a}{x-b}\right|.$$

E-mail address: dangvugiang@yahoo.com.

http://dx.doi.org/10.1016/j.jat.2015.08.001
0021-9045/

To compute the Hilbert transform of several functions we define the complex Hardy spaces $\mathfrak{H}^p(\mathbb{C}_+)$ where $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $1 \le p < \infty$. More exactly, $\varphi \in \mathfrak{H}^p(\mathbb{C}_+)$ [4] if φ is analytic in $\mathbb{C}_+$ and

$$\|\varphi\|_p^p := \sup_{y>0} \int_{-\infty}^{\infty} |\varphi(x+iy)|^p\, dx < \infty.$$

It is well known that if $\varphi \in \mathfrak{H}^p(\mathbb{C}_+)$ then for almost every $x \in \mathbb{R}$ there is $\lim_{y\to 0} \varphi(x+iy) =: f(x) + i\tilde{f}(x)$, where $f, \tilde{f} \in L^p(\mathbb{R})$ if $1 < p < \infty$. Note that $\tilde{f}(x) = \mathrm{Re}\varphi(x+i0)$ for $f(x) = -\mathrm{Im}\varphi(x+i0)$. Therefore, the Hilbert transform is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ [4] and $H(Hf) = -f$ for every $f \in L^p(\mathbb{R})$ with $1 < p < \infty$. Moreover,

$$\int_{-\infty}^{\infty} f(x)\,\tilde{g}(x)\, dx = -\int_{-\infty}^{\infty} \tilde{f}(x)\, g(x)\, dx$$

for $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ with $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Replace g by $\chi_{(a,b)}$ we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \ln \left| \frac{x-a}{x-b} \right| dx = -\int_a^b \tilde{f}(x)\, dx$$

for every $f \in L^p(\mathbb{R})$. For a compactly supported function $f \in L^p(\mathbb{R})$ we can define the logarithmic integral

$$F(b) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \ln \frac{1}{|x-b|} \cdot dx.$$

Then

$$F(b) - F(a) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \ln \left| \frac{x-a}{x-b} \right| dx = -\int_a^b \tilde{f}(x)\, dx.$$

Hence, F is locally absolutely continuous with weak derivative $-\tilde{f}$. Specially, we have

Theorem 1. *If a function $f \in L^p$ $(p > 1)$ is supported in a set E of finite disjoint compact intervals and the logarithmic integral of f is constant in E then $\tilde{f} = 0$ in E.*

Let $\varphi \in \mathfrak{H}^p(\mathbb{C}_+)$ and $\phi \in \mathfrak{H}^q(\mathbb{C}_+)$ with $\frac{1}{p} + \frac{1}{q} \le 1$. Then $\varphi\phi \in \mathfrak{H}^r(\mathbb{C}_+)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ so we have

$$H\left(f\tilde{g} + \tilde{f}g\right) = \tilde{f}\tilde{g} - fg \quad \text{with } f \in L^p(\mathbb{R}) \text{ and } g \in L^q(\mathbb{R}). \tag{1.1}$$

Finally, let $a_1 < a_2 < \cdots < a_{2\ell}$,

$$E = \bigcup_{k=1}^{\ell} \left[a_{2k-1}, a_{2k}\right] \quad \text{and} \quad K(x) = \prod_{j=1}^{2\ell} \left(x - a_j\right).$$

Then $K(x) \le 0$ if and only if $x \in E$. Let

$$g(x) = g_E(x) = \begin{cases} (-1)^{\ell-k}\sqrt{|K(x)|} & \text{if } x \in \left[a_{2k-1}, a_{2k}\right] \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$