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Notes on (s, t)-weak tractability: A refined classification of problems with (sub)exponential information complexity

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Abstract

In the last 20 years a whole hierarchy of notions of tractability was proposed and analyzed by several authors. These notions are used to classify the computational hardness of continuous numerical problems $S = (S_d)_{d\in\mathbb{N}}$ in terms of the behavior of their information complexity $n(\varepsilon, S_d)$ as a function of the accuracy ε and the dimension d. By now a lot of effort was spent on either proving quantitative positive results (such as, e.g., the concrete dependence on ε and d within the well-established framework of polynomial tractability), or on qualitative negative results (which, e.g., state that a given problem suffers from the so-called curse of dimensionality). Although several weaker types of tractability were introduced recently, the theory of information-based complexity still lacks a notion which allows to quantify the exact (sub-/super-) exponential dependence of $n(\varepsilon, S_d)$ on both parameters ε and d. In this paper we present the notion of (s, t)-weak tractability which attempts to fill this gap. Within this new framework the parameters s and t are used to quantitatively refine the huge class of polynomially intractable problems. For linear, compact operators between Hilbert spaces we provide characterizations of (s, t)-weak tractability w.r.t. the worst case setting in terms of singular values. In addition, our new notion is illustrated by classical examples which recently

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attracted some attention. In detail, we study approximation problems between periodic Sobolev spaces and integration problems for classes of smooth functions.

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1. Introduction

Let $S = (S_d)_{d\in\mathbb{N}}$ denote a multivariate numerical problem, i.e., a sequence of *solution operators* S_d, where each of them maps problem elements f from a subset of some normed (*source*) space F_d onto its solution $S_d(f)$ in some other (*target*) space G_d. In the following we refer to the parameter d as the *dimension* of the problem instance S_d. Typical examples cover approximation problems (where S_d is an embedding operator between spaces of d-variate functions) or integration problems (where $S_d(f)$ is defined as the integral of f over some d-dimensional domain).

We are interested in the computational hardness of S with respect to given classes of algorithms. This can be modeled by the *information complexity* $n(\varepsilon, S_d)$ which is defined as the minimal number of information operations that are needed to solve the d-dimensional problem with accuracy $\varepsilon > 0$:

$$n^{\mathrm{abs}}(\varepsilon, S_d) := \min\{n \in \mathbb{N}_0 \mid e(n,d) \leq \varepsilon\}. \tag{1}$$

Therein the quantity $e(n, d)$ is defined as the minimal error (measured w.r.t. a given *setting*) that can be achieved among all algorithms (within the class under consideration) that use at most $n \in \mathbb{N}_0$ information operations (degrees of freedom) on the input f to approximate the exact solution $S_d(f)$. The *initial error* of the d-dimensional problem instance S_d is denoted by

$$\varepsilon_d^{\mathrm{init}} := e(0,d), \quad d \in \mathbb{N}.$$

Besides the information complexity with respect to the *absolute error criterion* as defined in (1) we also consider the respective quantity w.r.t. the *normalized error criterion*,

$$n^{\mathrm{norm}}(\varepsilon, S_d) := \min\left\{n \in \mathbb{N}_0 \mid e(n,d) \leq \varepsilon \cdot \varepsilon_d^{\mathrm{init}}\right\}, \tag{2}$$

which measures how many pieces of information are needed to reduce the initial error by some factor $\varepsilon \in (0, 1]$. Typical classes of algorithms under consideration are, e.g., methods based on arbitrary linear functionals (information in Λ^{all}), or algorithms which are allowed to use function values (Λ^{std}) only. Moreover, one may stick to linear methods only, allow or prohibit adaption and/or randomization. Possible settings include the *worst case*, *average case*, *probabilistic*, and the *randomized setting*. For concrete definitions, explicit complexity results, and further references, see, e.g., the monographs [10–12,16], as well as the recent survey [18].

For the ease of presentation, in what follows, we mainly focus our attention on linear algorithms and their worst case errors among the unit ball $\mathcal{B}(F_d) := \{f \in F_d \mid \|f \mid F_d\| \leq 1\}$ of our source space F_d. That is, we set

$$e(n,d) := e^{\mathrm{wor}}(n,d;\Lambda) := \inf_{A_{n,d}} \sup_{f\in\mathcal{B}(F_d)} \left\| S_d(f) - A_{n,d}(f) \mid G_d \right\|,$$