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Journal of Approximation Theory 198 (2015) 63–110

JOURNAL OF Approximation Theory

www.elsevier.com/locate/jat

Full length article

Asymptotics of determinants of Hankel matrices via non-linear difference equations

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Received 7 January 2014; received in revised form 9 April 2015; accepted 13 May 2015
Available online 12 June 2015

Communicated by Arno B J Kuijlaars

Abstract

E. Heine in the 19th century studied a system of orthogonal polynomials associated with the weight $[x(x-\alpha)(x-\beta)]^{-\frac{1}{2}}$, $x \in [0, \alpha]$, $0 < \alpha < \beta$. A related system was studied by C. J. Rees in 1945, associated with the weight $\left[(1-x^2)(1-k^2x^2)\right]^{-\frac{1}{2}}$, $x \in [-1, 1]$, $k^2 \in (0, 1)$. These are also known as elliptic orthogonal polynomials, since the moments of the weights may be expressed in terms of elliptic integrals. Such orthogonal polynomials are of great interest because the corresponding Hankel determinant, depending on a parameter k^2, where $0 < k^2 < 1$ is the τ function of a particular Painlevé VI, the special cases of which are related to enumerative problems arising from string theory. We show that the recurrence coefficients, denoted by $\beta_n(k^2)$, $n = 1, 2, \ldots$; and $\mathsf{p}_1(n, k^2)$, the coefficients of x^{n-2} of the monic polynomials orthogonal with respect to a generalized version of the weight studied by Rees,

$$(1-x^2)^{\alpha}(1-k^2x^2)^{\beta}, \quad x \in [-1, 1], \ \alpha > -1, \ \beta \in \mathbb{R},$$

satisfy second order non-linear difference equations. The large n expansion based on the difference equations when combined with known asymptotics of the leading terms of the associated Hankel determinant yields a complete asymptotic expansion of the Hankel determinant. The Painlevé equation

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http://dx.doi.org/10.1016/j.jat.2015.05.002

is also discussed as well as the generalization of the linear second order differential equation found by Rees.

MSC: 15B52; 33C47; 34E05; 42C05; 47B35

Keywords: Hankel determinants; Elliptic orthogonal polynomials; Asymptotic expansions; Non-linear difference equations; Painlevé equations; Random matrix theory

1. Introduction

The study of Hankel determinants has seen a flurry of activity in recent years in part due to connections with Random Matrix theory (RMT). This is because Hankel determinants compute the most fundamental objects studied in RMT. For example, the determinants may represent the partition function for a particular random matrix ensemble, or they might be related to the distribution of the largest eigenvalue or they may represent the generating function for a random variable associated to the ensemble.

Another recent interesting application of Hankel determinants is to compute certain Hilbert series that are used to count the number of gauge invariant quantities on moduli spaces and to characterize moduli spaces of a wide range of supersymmetric gauge theories. Many aspects of supersymmetric gauge theories can be analyzed exactly, providing a “laboratory” for the dynamic of gauge theories. For additional information about this topic, see [15,8]. In these papers, heavy use are made of the mathematics involving two important types of matrices: Toeplitz and Hankel.

Often there is an associated Painlevé equation that is satisfied by the logarithm of the Hankel determinant with respect to some parameter. This is true, for example, in the Gaussian Unitary ensemble and for many other classical cases [32]. In a recent development, one finds that Painlevé equations also appear in the information theoretic aspect of wireless communication systems [17]. Once the Painlevé equation is found, then the Hankel determinant is much better understood. Asymptotics can be found via the Painlevé equation, scalings can be made to find limiting densities, and in general the universal nature of the distributions can be analyzed. Other methods, including Riemann–Hilbert techniques and general Fredholm operator theory methods, have also been used very successfully to find these asymptotics along with the Painlevé equation analysis.

In this paper, where our approach is different, our focus is on the modified Jacobi weight,

$$(1-x^2)^{\alpha}(1-k^2x^2)^{\beta}, \quad x \in [-1,1],\ \alpha > -1,\ \beta \in \mathbb{R},\ k^2 \in (0,1).$$

We find asymptotics for the determinant, but our main technique is to compute these asymptotics from difference equations and then combine the information obtained from the difference equation with known asymptotics for the leading order terms. This is done by finding equations for auxiliary quantities defined by the corresponding orthogonal polynomials. The main idea is to use the very useful and practical ladder operator approach developed in [11] and [12]. Recent applications of Riemann–Hilbert techniques on the asymptotics of orthogonal polynomials and Hankel determinants associated with similar modified Jacobi weights can be found in [33,34].