Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 195 (2015) 1–18

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Spherical designs of harmonic index t

Eiichi Bannai[a], Takayuki Okuda[b], Makoto Tagami[c,*]

[a] Department of Mathematics, Shanghai Jiao Tong University, 800 Dongchuan RD, Minhang District, Shanghai, China
[b] Graduate School of Information Sciences, Tohoku University, Aramaki aza Aoba 6-3-09, Aoba-ku Sendai-city Miyagi-pref. 980-8579, Japan
[c] Graduate School of Computer Science and Systems Engineering, Kyushu Institute of Technology, 680-4 Kawazu, Iizuka-shi, Fukuoka, 820-8502, Japan

Received 8 September 2013; received in revised form 13 May 2014; accepted 25 June 2014
Available online 3 July 2014

Communicated by Special Issue Guest Editor

Dedicated to Richard Askey for his 80th birthday

Abstract

Spherical t-design is a finite subset on sphere such that, for any polynomial of degree at most t, the average value of the integral on sphere can be replaced by the average value at the finite subset. It is well-known that an equivalent condition of spherical design is given in terms of harmonic polynomials. In this paper, we define a spherical design of harmonic index t from the viewpoint of this equivalent condition, and we give its construction and a Fisher type lower bound on the cardinality. Also we investigate whether there is a spherical design of harmonic index attaining the bound.

Keywords: Spherical design of harmonic index; Fisher type inequality; Equiangular lines

1. Introduction

Let t be a natural number, S^{n-1} the $(n-1)$-dimensional unit sphere centered at the origin. A finite nonempty subset X on S^{n-1} is called a *spherical t-design* if, for any polynomial

* Corresponding author.
E-mail addresses: bannai@sjtu.edu.cn, bannai@math.kyushu-u.ac.jp (E. Bannai), okuda@ims.is.tohoku.ac.jp (T. Okuda), tagami77@gmail.com, tagami@ces.kyutech.ac.jp (M. Tagami).

http://dx.doi.org/10.1016/j.jat.2014.06.010
0021-9045/

$f(x) = f(x_1, \ldots, x_n)$ of degree at most t, the following equality holds:

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(x)d\sigma(x) = \frac{1}{|X|}\sum_{x\in X} f(x),$$

where σ is an $O(\mathbb{R}^n)$-invariant measure on S^{n-1} and $|S^{n-1}|$ denotes the surface volume of the sphere S^{n-1}. The concept of spherical design was defined by Delsarte–Goethals–Seidel (refer to [8,7,2,3]). A spherical t-design means to be a good configuration of points on sphere so that the average value of the integral of any polynomial of degree at most t on sphere can be replaced by the average value at the finite set on sphere.

Let $\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ be the Laplacian. A polynomial $f(x)$ is *harmonic* if $\triangle f(x) = 0$. Then put

$$\mathrm{Harm}_t(\mathbb{R}^n) = \{f(x) \mid f(x) \text{ is a harmonic and homogeneous polynomial of degree } t \text{ on } \mathbb{R}^n\}.$$

The dimension of $\mathrm{Harm}_t(\mathbb{R}^n)$ is $\binom{n+t-1}{t} - \binom{n+t-3}{t-2}$ (refer to [1, page 478]). Some equivalent conditions of spherical design are known. In particular, the following condition is quite often used [7,2,3]:

$$\text{For any } f(x) \in \mathrm{Harm}_j(\mathbb{R}^d) \quad \text{and} \quad 1 \le j \le t, \qquad \sum_{x\in X} f(x) = 0.$$

From this condition, we introduce the following notion which is the main subject in this paper:

Definition 1.1 (*Spherical Design of Harmonic Index t*)**.** A finite nonempty subset X on S^{n-1} is called a *spherical design of harmonic index t* (or simply, harmonic index t-design) if, for any $f(x) \in \mathrm{Harm}_t(\mathbb{R}^n)$, $\sum_{x\in X} f(x) = 0$.

We are interested in what figure appears as spherical designs of harmonic index, and whether we can give a natural lower bound for harmonic index designs similar to the case of usual spherical design.

When t is odd, any antipodal two points $X = \{x, -x\}$ forms a harmonic index t-design because $\sum_{x\in X} f(x) = f(x) + f(-x) = f(x) - f(x) = 0$ for any $f(x)$ in $\mathrm{Harm}_t(\mathbb{R}^n)$. So, from now on, we consider only the case when t is even.

When t is even, for any $f(x) \in \mathrm{Harm}_t(\mathbb{R}^n)$, $f(-x) = f(x)$. So we remark that one can take harmonic index t-designs just on hemisphere. For any $n, t \in \mathbb{N}$, from Seymour–Zaslavsky's theorem [15], if we make the number of vertices big enough, there always exists a harmonic index t-design on S^{n-1}. We denote the minimum cardinality of harmonic index t-design on S^{n-1} by $A(n, t)$. From the above, we see that, when t is odd, $A(n, t) = 2$.

First we consider the case when $n = 2$ and $t = 2e$. Let x, y be two unit vectors in $\mathbb{R}^2$ with angular $\theta = j\pi/2e$ for odd j. Then $X = \{x, y\}$ is a harmonic index $2e$-design on S^1. So we see that $A(2, t) = 2$.

Next we consider the case when $t = 2$ and $n \ge 2$. Let $X = \{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ (that is, an antipodal half part of regular cross-polytope). Then it is easy to see that X is a harmonic index 2-design on S^{n-1}. Therefore $A(n, 2) \le n$. In fact, we will show $A(n, 2) = n$ later.[1]

[1] Also we note that this fact is shown as follows, too: 3-designs contain a basis of the space, so have cardinality $\ge n$.