Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 195 (2015) 57–69
Journal of Approximation Theory
www.elsevier.com/locate/jat

Full length article

Positivity of rational functions and their diagonals

Armin Straub[a,1], Wadim Zudilin[b,*]

[a] Max-Planck-Institut für Mathematik, Vivatsgasse 7, Bonn D-53111, Germany
[b] School of Mathematical and Physical Sciences, The University of Newcastle, Callaghan NSW 2308, Australia

Received 16 September 2013; received in revised form 26 April 2014; accepted 14 May 2014
Available online 28 May 2014

Communicated by Special Issue Guest Editor

To Dick Askey on the occasion of his birthday, with many positive wishes

Abstract

The problem to decide whether a given rational function in several variables is positive, in the sense that all its Taylor coefficients are positive, goes back to Szegő as well as Askey and Gasper, who inspired more recent work. It is well known that the diagonal coefficients of rational functions are D-finite. This note is motivated by the observation that, for several of the rational functions whose positivity has received special attention, the diagonal terms in fact have arithmetic significance and arise from differential equations that have modular parametrization. In each of these cases, this allows us to conclude that the diagonal is positive.

Further inspired by a result of Gillis, Reznick and Zeilberger, we investigate the relation between positivity of a rational function and the positivity of its diagonal.
Crown Copyright © 2014 Published by Elsevier Inc. All rights reserved.

Keywords: Positivity; Rational function; Hypergeometric function; Modular function; Apéry-like sequence; Multivariate asymptotics

* Corresponding author.
E-mail address: wzudilin@gmail.com (W. Zudilin).
URLs: http://arminstraub.com/ (A. Straub), http://wain.mi.ras.ru/ (W. Zudilin).
[1] Current address: Department of Mathematics, University of Illinois, 1409 West Green St, Urbana, IL 61801, USA.

http://dx.doi.org/10.1016/j.jat.2014.05.012
0021-9045/Crown Copyright © 2014 Published by Elsevier Inc. All rights reserved.

1. Introduction

The question to decide whether a given rational function is positive, that is, whether its Taylor coefficients are all positive, goes back to Szegő [25] and has since been investigated by many authors including Askey and Gasper [3–5], Koornwinder [17], Ismail and Tamhankar [14], Gillis, Reznick and Zeilberger [11], Kauers [15], Straub [24], Kauers and Zeilberger [16], Scott and Sokal [23]. The interested reader will find a nice historical account in [23]. A particularly interesting instance is the Askey–Gasper rational function

$$A(x, y, z) := \frac{1}{1 - x - y - z + 4xyz}, \tag{1}$$

whose positivity is proved in [5,11]. Generalizations to more than three variables are rarely tractable, with the longstanding conjecture of the positivity of

$$\frac{1}{1 - x - y - z - w + \frac{2}{3}(xy + xz + xw + yz + yw + zw)}, \tag{2}$$

also referred to as the Lewy–Askey problem. Very recently, Scott and Sokal [23] succeeded in proving the non-negativity of (2), both in an elementary way by an explicit Laplace-transform formula and based on more general results on the basis generating polynomials of certain classes of matroids. Note that by a result from [16] the positivity of (2) would follow from the positivity of

$$D(x, y, z, w) := \frac{1}{1 - x - y - z - w + 2(yzw + xzw + xyw + xyz) + 4xyzw}, \tag{3}$$

which is still an open problem. In another direction, Gillis, Reznick and Zeilberger conjecture in [11] that

$$\frac{1}{1 - (x_1 + x_2 + \cdots + x_d) + d!x_1x_2 \cdots x_d} \tag{4}$$

has non-negative coefficients for any $d \geqslant 4$ (this is false for $d = 2, 3$). It is further asserted (though the proof is "omitted due to its length") that, in order to show the non-negativity of the rational functions in (4), it suffices to prove that their *diagonal* Taylor coefficients are non-negative. Modulo this claim, the cases $d = 4, 5, 6$ were established by Kauers [15], who found and examined recurrences for the respective diagonal coefficients.

The above claim from [11] suggests the following question. Here, we denote by $e_k(x_1, \ldots, x_d)$ the elementary symmetric polynomials defined by

$$\prod_{j=1}^{d}(x + x_j) = \sum_{k=0}^{d} e_k(x_1, \ldots, x_d)x^{d-k}. \tag{5}$$

Question 1.1. Under what (natural) condition(s) is the positivity of a rational function $h(x_1, \ldots, x_d)$ of the form

$$h(x_1, \ldots, x_d) = \frac{1}{\sum_{k=0}^{d} c_k e_k(x_1, \ldots, x_d)} \tag{6}$$

implied by the positivity of its diagonal? For example, would the positivity of $h(x_1, \ldots, x_{d-1}, 0)$ be a sufficient condition?

Download English Version:

https://daneshyari.com/en/article/4606947

Download Persian Version:

https://daneshyari.com/article/4606947

Daneshyari.com