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Differential equations for discrete Laguerre–Sobolev orthogonal polynomials☆

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Abstract

The aim of this paper is to study differential properties of orthogonal polynomials with respect to a discrete Laguerre–Sobolev bilinear form with mass point at zero. In particular we construct the orthogonal polynomials using certain Casorati determinants. Using this construction, we prove that they are eigenfunctions of a differential operator (which will be explicitly constructed). Moreover, the order of this differential operator is explicitly computed in terms of the matrix which defines the discrete Laguerre–Sobolev bilinear form.

Keywords: Orthogonal polynomials; Differential operators; Laguerre polynomials; Discrete Laguerre–Sobolev orthogonal polynomial

1. Introduction and results

The issue of orthogonal polynomials (with respect to a positive measure) which are also common eigenfunctions of a second order differential operator goes back at least for two centuries, when Legendre introduced the first family of what we call today classical orthogonal polynomials. As S. Bochner established in 1929 [3], there are only three families of classical

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orthogonal polynomials: Hermite, Laguerre and Jacobi (and Bessel polynomials if signed measures are considered).

H.L. Krall raised in 1939 [17,18] the problem of finding orthogonal polynomials which are also common eigenfunctions of a higher order differential operator with polynomial coefficients. He obtained a complete classification for the case of a differential operator of order four [18]. Besides the classical families of Hermite, Laguerre and Jacobi (satisfying second order differential equations), he found three other families of orthogonal polynomials which are also eigenfunctions of a fourth order differential operator. One of them is orthogonal with respect to a positive measure which consists of a Laguerre weight together with a Dirac delta at the end point of its interval of orthogonality: $e^{-x} + M_0\delta_0$.

Forty years later, L.L. Littlejohn [19,20] discovered new families satisfying sixth and eighth order differential equations, respectively. They are orthogonal with respect to

$$x^\alpha e^{-x} + M_0\delta_0, \quad x > 0, \tag{1.1}$$

with $\alpha = 1, 2$, respectively. The general result for α a nonnegative integer was proved by J. Koekoek and R. Koekoek who showed in 1991 that orthogonal polynomials with respect to (1.1) are also eigenfunctions of an infinite order differential operator, except for nonnegative integer values of α for which the order reduces to $2\alpha + 4$ [14].

Some years later discrete Laguerre–Sobolev orthogonal polynomials which are also common eigenfunctions of a higher order differential operator entered into the picture. R. Koekoek and H. G. Meijer [16] introduced orthogonal polynomials with respect to the discrete Laguerre–Sobolev inner product

$$\langle p, q\rangle = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + M_0 p(0)q(0) + M_1 p'(0)q'(0), \quad M_0 \geq 0, M_1 > 0,$$

and later on R. Koekoek [12,13] found that for $\alpha = 0, 1, 2$, those orthogonal polynomials are also eigenfunctions of a differential operator with polynomial coefficients of order $2\alpha + 8$ when $M_0 = 0$ and $4\alpha + 10$ for $M_0 > 0$. This result was soon extended for nonnegative integers α by J. Koekoek, R. Koekoek and H. Bavinck [15]. Using a different approach, P. Iliev [9] has recently extended these results for a Laguerre–Sobolev inner product of the form

$$\langle p, q\rangle = \int_0^\infty p(x)q(x)x^{\alpha-2}e^{-x}dx + (p(0), p'(0)) \begin{pmatrix} M_{0,0} & M_{0,1} \\ M_{0,1} & M_{1,1} \end{pmatrix} \begin{pmatrix} q(0) \\ q'(0) \end{pmatrix}.$$

(For other related papers see [10] and [11]).

For $\alpha \neq -1, -2, \ldots$, denote by $\mu_\alpha(x)$ the orthogonalizing weight for the Laguerre polynomials. Only when $\alpha > -1$, $\mu_\alpha(x)$, $x > 0$, is positive, and then

$$\mu_\alpha(x) = x^\alpha e^{-x}, \quad x > 0. \tag{1.2}$$

Let M be a $m \times m$ matrix. The purpose of this paper is to prove in a constructive way that if α and m are positive integers with $\alpha \geq m$, then the orthogonal polynomials with respect to a discrete Laguerre–Sobolev bilinear form

$$\langle p, q\rangle = \int_0^\infty p(x)q(x)\mu_{\alpha-m}(x)dx + (p(0), \ldots, p^{(m-1)}(0))M \begin{pmatrix} q(0) \\ \vdots \\ q^{(m-1)}(0) \end{pmatrix},$$

are eigenfunctions of a differential operator with polynomial coefficients.