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Sums of monomials with large Mahler measure

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Abstract

For $n \geq 1$ let

$$\mathcal{A}_n := \left\{ P : P(z) = \sum_{j=1}^{n} z^{k_j} : 0 \leq k_1 < k_2 < \cdots < k_n, k_j \in \mathbb{Z} \right\},$$

that is, $\mathcal{A}_n$ is the collection of all sums of n distinct monomials. These polynomials are also called Newman polynomials. If $\alpha < \beta$ are real numbers then the Mahler measure $M_0(Q, [\alpha, \beta])$ is defined for bounded measurable functions $Q(e^{it})$ on $[\alpha, \beta]$ as

$$M_0(Q, [\alpha, \beta]) := \exp\left(\frac{1}{\beta - \alpha} \int_\alpha^\beta \log |Q(e^{it})| dt \right).$$

Let $I := [\alpha, \beta]$. In this paper we examine the quantities

$$L_n^0(I) := \sup_{P \in \mathcal{A}_n} \frac{M_0(P, I)}{\sqrt{n}} \quad \text{and} \quad L^0(I) := \liminf_{n \to \infty} L_n^0(I)$$

with $0 < |I| := \beta - \alpha \leq 2\pi$.

Keywords: Large sieve inequalities; Mahler measure; L_1 norm; Constrained coefficients; Fekete polynomials; Littlewood polynomials; Newman polynomials; Sums of monomials

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1. Introduction

The large sieve of number theory [34] asserts that if

$$P(z) = \sum_{k=-n}^{n} a_k z^k$$

is a trigonometric polynomial of degree at most n,

$$0 \le t_1 < t_2 < \cdots < t_m \le 2\pi,$$

and

$$\delta := \min\{t_1 - t_0, t_2 - t_1, \ldots, t_m - t_{m-1}\}, \quad t_0 := t_m - 2\pi,$$

then

$$\sum_{j=1}^{m} |P(e^{it_j})|^2 \le \left(\frac{n}{2\pi} + \delta^{-1}\right) \int_0^{2\pi} |P(e^{it})|^2 \, dt.$$

There are numerous extensions of this to the L_p norm (or involving $\psi(|P(e^{it})|^p)$, where ψ is a convex function), $p > 0$, and even to subarcs. See [23,32]. There are versions of this that estimate Riemann sums, for example, with $t_0 := t_m - 2\pi$,

$$\sum_{j=1}^{m} |P(e^{it_j})|^2 (t_j - t_{j-1}) \le C \int_0^{2\pi} |P(e^{it})|^2 \, dt,$$

with a constant C depending only on n/m but independent of P and $\{t_1, t_2, \ldots, t_m\}$. These are often called forward Marcinkiewicz–Zygmund inequalities. Converse Marcinkiewicz–Zygmund inequalities provide estimates for the integrals above in terms of the sums on the left-hand side, see [29,31,33,37]. A particularly interesting case is that of the L_0 norm. A result in [19] asserts that if $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ are the n-th roots of unity, and P is a polynomial of degree at most n, then

$$\prod_{j=1}^{n} |P(\alpha_j)|^{1/n} \le 2M_0(P), \tag{1.1}$$

where

$$M_0(P) := \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{it})| \, dt\right)$$

is the Mahler measure of P. In [19] we were focusing on showing that methods of subharmonic function theory provide a simple and direct way to generalize previous results. We also extended (1.1) to points other than the roots of unity and exponentials of logarithmic potentials of the form

$$P(z) = c \, \exp\left(\int \log |z - t| \, d\nu(t)\right),$$

where $c \ge 0$ and ν is a positive Borel measure of compact support with $\nu(\mathbb{C}) \ge 0$. Inequalities for exponentials of logarithmic potentials and generalized polynomials were studied by several authors, see [5,13,14,18–21], for instance.