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Basis partition polynomials, overpartitions and the Rogers–Ramanujan identities

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Dedicated with admiration to my friend, Richard Askey

Abstract

In this paper, a common generalization of the Rogers–Ramanujan series and the generating function for basis partitions is studied. This leads naturally to a sequence of polynomials, called BsP-polynomials. In turn, the BsP-polynomials provide simultaneously a proof of the Rogers–Ramanujan identities and a new, more rapidly converging series expansion for the basis partition generating function. Finally the basis partitions are identified with a natural set of overpartitions.

Keywords: Basis partitions; Overpartitions; Rogers–Ramanujan identities

1. Introduction

The late Hansraj Gupta [8] introduced the concept of basis partitions. Basis partitions are defined in terms of successive ranks [6] or the "rank vector" of a partition.

Namely, each partition, π, of a positive integer contains a largest square of nodes in its Ferrers graph. This square is called the Durfee square. If the Durfee square has side d, we define the ith rank r_i of π $(1 \leq i \leq d)$ as the difference between the number of nodes in the ith row of the Ferrers graph of π and the number in the ith column. The rank vector for π is $(r_1, r_2, \ldots, r_d)$.

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For example, if π is the partition $5+5+4+2+2+1$, then it Ferrers graph is:

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Its rank vector is $(-1, 0, 1)$.

Gupta [8] showed that for every rank vector, $\vec{r}$, there is a smallest integer that has a partition with rank vector $\vec{r}$, and that partition is unique. This partition is called the basis partition for $\vec{r}$. We let $B(n)$ denote the number of basis partitions of n.

For example, the basis partition for $(-1, 0, 1)$ is $4+4+4+2+1$.

In [11], Nolan, Savage and Wilf showed that

$$\sum_{n=0}^{\infty} B(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q)_n}{(q;q)_n}, \tag{1.1}$$

where

$$(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}). \tag{1.2}$$

Hirschhorn [10] gave a new proof of (1.1) and related basis partitions to the Rogers–Ramanujan series from the first Rogers–Ramanujan identity [4, p. 113]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}. \tag{1.3}$$

Our central object is to study

$$G(a,x;q) := \sum_{n=0}^{\infty} \frac{a^n q^{n^2}(x;q)_n}{(q;q)_n}. \tag{1.4}$$

Following the work of Nola, Savage and Wilf [11] and of Hirschhorn [10], Alladi had in 2007 considered $G(1,-zq;q)$ and had interpreted the power of z as representing the signature of a basis partition (namely the number of different parts below the Durfee square); he then studied basis partitions combinatorially [1] with emphasis on the signature.

Notice that if we set $x=-q$ and set $a=1$ in (1.4), we get the series in (1.1), and if we set $x=0$ and $a=1$ we get the series in (1.3). We want to find an identity for $G(a,x;q)$ which both leads directly to the Rogers–Ramanujan identities and also provides a new representation of the series in (1.1).

We shall prove

Theorem 1.

$$G(a,x;q) = \frac{1}{(aq;q)_\infty}\left(1+\sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}(1-aq^{2n})(-1)^n q^{n(3n-1)/2} a^n B_n(a,x)}{(q;q)_n}\right) \tag{1.5}$$