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# Explicit matrix inverses for lower triangular matrices with entries involving Jacobi polynomials

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**Abstract**

For a two-parameter family of lower triangular matrices with entries involving Jacobi polynomials an explicit inverse is given, with entries involving a sum of two Jacobi polynomials. The formula simplifies in the Gegenbauer case and then one choice of the parameter solves an open problem in a recent paper by Koelink, van Pruijssen & Román. The two-parameter family is closely related to two two-parameter groups of lower triangular matrices, of which we also give the explicit generators. Another family of pairs of mutually inverse lower triangular matrices with entries involving Jacobi polynomials, unrelated to the family just mentioned, was given by J. Koekoek & R. Koekoek (1999). We show that this last family is a limit case of a pair of connection relations between Askey–Wilson polynomials having one of their four parameters in common.


*Keywords:* Jacobi polynomials; Gegenbauer polynomials; Inverses of explicit lower triangular matrices; Biorthogonal systems; Askey–Wilson polynomials; Connection coefficients

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## 1. Introduction

This note started as a kind of supplement to the paper [13] by Koelink, van Pruijssen and Román, but gradually it got a wider scope. As for [13] it solves an open problem there (see Theorem 2.1 and paragraph after Theorem 6.2 in [13]) to invert a lower triangular matrix with entries involving Gegenbauer polynomials. For a two-parameter family of such matrices involving Jacobi polynomials we give the explicit inverse matrix in Theorem 4.1. Specialization to Gegenbauer polynomials then gives a one-parameter family. One specialization of the parameter in the latter family gives the inversion desired in [13]. Another specialization gives a matrix inversion already handled by Brega and Cagliero [3].

Our two-parameter family of Jacobi polynomials is closely related to two commutative two-parameter groups of lower triangular matrices involving Jacobi polynomials. We also give the explicit infinitesimal generators of these two-parameter groups. Furthermore we obtain a biorthogonality relation for two explicit systems of functions on $\mathbb{Z}$ involving Jacobi polynomials with respect to an explicit bilinear form on $\mathbb{Z}$.

Another two-parameter family of pairs of mutually inverse lower triangular matrices with entries involving Gegenbauer polynomials, unrelated to the family mentioned above, is implied by Brown and Roman [4, (4.14)]. J. Koekoek and R. Koekoek [11, (17)], unaware of [4], generalized a one-parameter subfamily of this two-parameter family to entries involving Jacobi polynomials. We will show that this last family can be realized as a limit case of a pair of connection relations between Askey–Wilson polynomials having one of their four parameters in common. These Askey–Wilson connection coefficients were first given by Askey and Wilson [2, (6.5)]. The limit case connects Jacobi polynomials $P_n^{(\alpha,\beta)}$ with shifted monomials $x \mapsto (x-y)^k$.

The contents of the paper are as follows. In Section 2 some preliminaries about Jacobi polynomials are given. Degenerate cases of Jacobi polynomials are classified in Section 3. The main results about the mutually inverse lower triangular matrices are stated in Section 4. This section ends with some open problems. The computations leading to the explicit inverse matrix of the first family of lower triangular matrices are given in Section 5. The two-parameter groups and their generators are treated in Section 6. The biorthogonal systems with respect to an explicit bilinear form are the topic of Section 7. Finally, the computations giving the limit of the Askey–Wilson connection relations are done in Section 8.

The reader may start in Section 4 and then continue with Section 5 or with Sections 6 and 7 or with Section 8. The preliminary Sections 2 and 3 can be consulted when needed.

## 2. Preliminaries about Jacobi polynomials

Jacobi polynomials (see for instance [15, Chapter IV], [1, Chapter 6], [8, Chapter 4], [12, Section 9.8], [14, Chapter 18]) can be expressed in terms of the Gauss hypergeometric function by

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &:= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix};\tfrac12(1-x)\right)\\ &= \sum_{k=0}^{n}\frac{(n+\alpha+\beta+1)_k\,(\alpha+k+1)_{n-k}}{k!\,(n-k)!}\left(\frac{x-1}{2}\right)^k. \end{aligned} \tag{2.1}$$

Note that they are well-defined for all values of $\alpha, \beta$. Their normalization avoids artificial singularities. Jacobi polynomials satisfy a Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\,(1-x)^{-\alpha}\,(1+x)^{-\beta}\left(\frac{d}{dx}\right)^n\Big((1-x)^{n+\alpha}(1+x)^{n+\beta}\Big). \tag{2.2}$$