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Full length article

A family of nonlinear difference equations: Existence, uniqueness, and asymptotic behavior of positive solutions☆

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Abstract

We study solutions $(x_n)_{n\in\mathbb{N}}$ of nonhomogeneous nonlinear second order difference equations of the type

$$\ell_n = x_n\left(\sigma_{n,1}\,x_{n+1} + \sigma_{n,0}\,x_n + \sigma_{n,-1}\,x_{n-1}\right) + \kappa_n\,x_n, \quad n \in \mathbb{N},$$
$$\text{with given initial data } \{x_0 \in \mathbb{R} \ \&\ x_1 \in \mathbb{R}^+\}$$

where

$$(\ell_n)_{n\in\mathbb{N}} \in \mathbb{R}^+ \ \&\ \left(\sigma_{n,0}\right)_{n\in\mathbb{N}} \in \mathbb{R}^+ \ \&\ (\kappa_n)_{n\in\mathbb{N}} \in \mathbb{R},$$

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and the left and right σ-coefficients satisfy either

$$\left(\sigma_{n,1}\right)_{n\in\mathbb{N}} \in \mathbb{R}^+ \ \&\ \left(\sigma_{n,-1}\right)_{n\in\mathbb{N}} \in \mathbb{R}^+$$

or

$$\left(\sigma_{n,1}\right)_{n\in\mathbb{N}} \in \mathbb{R}_0^+ \ \&\ \left(\sigma_{n,-1}\right)_{n\in\mathbb{N}} \in \mathbb{R}_0^+.$$

Depending on one's standpoint, such equations originate either from orthogonal polynomials associated with certain Shohat-Freud-type exponential weight functions or from Painlevé's discrete equation #1, that is, d-$\mathrm{P_I}$.

MSC: 39A22; 65Q10; 65Q30

Keywords: Nonhomogeneous nonlinear second order difference equations; Shohat-Freud-type exponential weight functions; Painlevé's discrete equation #1; Existence of solutions; Unicity of solutions; Asymptotic behavior

1. Preliminaries

Since the authors come from different cultures using different mathematical notation, we need to fix some of it right now in order to avoid subsequent misunderstanding.

The set of natural numbers $\mathbb{N}$ consists of all strictly positive integers. Furthermore, $\mathbb{R}^+ \stackrel{\text{def}}{=} \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_0^+ \stackrel{\text{def}}{=} \{x \in \mathbb{R} : x \geqslant 0\}$.

2. Introduction

This section will explain how the unlikely pair of JSZ and WVA became involved in this research via PN's manipulations. We justify its unusual length compared to the rest of the paper by the necessity of giving a proper historical perspective that will also serve as introduction for the subsequent papers that we plan to publish on nonlinear difference equations.

It was Géza Freud who brought the attention of the approximation theory and orthogonal polynomial communities to exponential weight functions with his extensive body of work in the 1970s that was suddenly interrupted by his untimely death in 1979 at the youthful age of 57 years.[1] In particular, Freud solved two special and, to some extent, simple cases of his *Freud conjectures* that, even today, are of extraordinary interest despite having been overshadowed by the incomparably deeper pathbreaking achievements by so many of us such as Alphonse Magnus, Evguenii A. Rakhmanov, Andrei A. Gonchar, Hrushikesh N. Mhaskar, Edward B. Saff, Doron S. Lubinsky, Vilmos Totik, and Guillermo López Lagomasino, in some kind of a chronological order.

The two special cases above refer to the asymptotic behavior of the recurrence coefficients in the three-term recurrence for the orthogonal polynomials associated with the weight functions $|x|^\rho \exp(-x^4)$ and $|x|^\rho \exp(-x^6)$ on $\mathbb{R}$ with $\rho > -1$, see [5]. In particular, [5, (23), p. 5] is the

[1] This statement is not entirely accurate; e.g., Mkhitar Djrbashian (aka Dzhrbashjan & Jerbashian) has a large body of work that is not that different from some of Freud's work but its international impact was negligible. In addition, exponential weights have long been of definite interest in areas such as the moment problem.