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Relations for Nielsen polylogarithms

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Abstract

Polylogarithms appear in many diverse fields of mathematics. Herein, we investigate relations amongst the restricted class of Nielsen-type (essentially, height one) polylogarithms, both generic and at special arguments including the sixth roots of unity. Numerical computations suggest that the collected relations, partially motivated by a previous study of the authors on log-sine integrals, are complete except in the case when the argument is the fundamental sixth root of unity. For use in other applications, all our results are implemented and accessible for use in symbolic computation or to facilitate numeric computation. In particular, the relations are explicitly exhibited in the case of low weights.

Keywords: Multiple polylogarithms; Clausen functions; Multiple zeta values; Log-sine integrals

1. Introduction and preliminaries

It is very well-known that the polylogarithmic sums

$$\mathrm{Gl}_{2k}(\tau) := \mathrm{Re}\,\mathrm{Li}_{2k}(e^{i\tau}) = \sum_{n=1}^{\infty} \frac{\cos(n\tau)}{n^{2k}} \tag{1}$$

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reduce to polynomials in τ. For instance,

$$\mathrm{Gl}_2(\tau) = \sum_{n=1}^{\infty} \frac{\cos(n\tau)}{n^2} = \frac{\tau^2}{4} - \frac{\pi\tau}{2} + \frac{\pi^2}{6}.$$

These are the simplest instances of (generic) relations among Clausen and Glaisher functions of Nielsen type (these terms will be defined below). In this note, we consider the entirety of these relations. Besides generic relations as above, we will be interested in additional relations at the special arguments $\pi/3$, $\pi/2$ and $2\pi/3$ because of their appearance in applications, see [8], both within mathematics (e.g., Apéry-like series for zeta values, multiple Mahler measure [9]) and physics (e.g., calculation of higher terms in ε-expansions of Feynman diagrams [13,15]). We should explicitly mention Broadhurst whose extensive work in [10] on more general polylogarithms in the sixth root of unity, see (29), is indicated in Remarks 3.9 and 4.2.

In Sections 2 and 3, we will review the structure of log-sine integrals, and then make the motivating observation that all generic relations amongst Clausen and Glaisher functions of Nielsen type are consequences of previous results on the evaluation of log-sine integrals. The classical evaluation of (1) as a polynomial may then be seen as the special case of the general fact that all Glaisher functions of odd depth (and Nielsen type) may be expressed in terms of Glaisher functions of lower depth.

In [8], it was shown that (generalized) log-sine integrals can be expressed in terms of polylogarithms of Nielsen type. Since tools for working with polylogarithmic functions are of practical utility – for instance, in physics – this conversion had been implemented in various computer algebra systems. However, for the (optional) purpose of simplifying results as much as possible, many reductions of multiple polylogarithms were built into our program [8]. In particular, a table of reductions at low weight was included. As a consequence of the results herein, we have been able to (almost completely) replace this table by general reductions which also work for higher weight. Numerical computations up to weight 10 suggest that our set of reductions is complete, except in the case of polylogarithms at the (fundamental) sixth root of unity for which, as discussed in Section 4.1, additional relations exist. This suggests that, even for the very restricted class of Nielsen-type polylogarithms, a deeper analysis of the relations at the sixth root of unity remains to be done.

1.1. Preliminaries

In the following, we will denote the *multiple polylogarithm* as studied for instance in [7] and [4, Chapter 3] by

$$\mathrm{Li}_{a_1,\ldots,a_k}(z) := \sum_{n_1>\cdots>n_k>0} \frac{z^{n_1}}{n_1^{a_1}\cdots n_k^{a_k}}. \tag{2}$$

For our purposes, the $a_1, \ldots, a_k$ will usually be positive integers and $a_1 \geq 2$ so that the sum converges for all $|z| \leq 1$. For example, $\mathrm{Li}_{2,1}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \sum_{j=1}^{k-1} \frac{1}{j}$. The usual notation will be used for repetitions so that, for instance, $\mathrm{Li}_{2,\{1\}^3}(z) = \mathrm{Li}_{2,1,1,1}(z)$. All such objects are generalizations of Euler's original dilogarithm $\mathrm{Li}_2(z)$ [2,20]. Note that $\mathrm{Li}_1(z) = -\log(1-z)$.

Moreover, *multiple zeta values* [4, Chapter 3] are denoted by

$$\zeta(a_1, \ldots, a_k) := \mathrm{Li}_{a_1,\ldots,a_k}(1).$$