Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 193 (2015) 99–127

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

The Rogers–Ramanujan continued fraction and its level 13 analogue

Shaun Cooper*, Dongxi Ye

Institute of Natural and Mathematical Sciences, Massey University-Albany, Private Bag 102904, North Shore Mail Centre, Auckland, New Zealand

Received 18 July 2013; received in revised form 14 January 2014; accepted 22 January 2014
Available online 6 February 2014

Communicated by Special Issue Guest Editor

Dedicated to Richard Askey in celebration of his 80th birthday

Abstract

One of the properties of the Rogers–Ramanujan continued fraction is its representation as an infinite product given by

$$r(q) = q^{1/5} \prod_{j=1}^{\infty} (1 - q^j)^{\left(\frac{j}{5}\right)}$$

where $\left(\frac{j}{p}\right)$ is the Legendre symbol. In this work we study the level 13 function

$$R(q) = q \prod_{j=1}^{\infty} (1 - q^j)^{\left(\frac{j}{13}\right)}$$

and establish many properties analogous to those for the fifth power of the Rogers–Ramanujan continued fraction. Many of the properties extend to other levels ℓ for which $\ell - 1$ divides 24, and a brief account of these results is included.

Keywords: Dedekind eta function; Eisenstein series; Hypergeometric function; Modular form; Pi; Ramanujan's theories of elliptic functions to alternative bases

* Corresponding author.
E-mail addresses: s.cooper@massey.ac.nz (S. Cooper), lawrencefrommath@gmail.com (D. Ye).

http://dx.doi.org/10.1016/j.jat.2014.01.008

1. Introduction

The Rogers–Ramanujan continued fraction $\mathfrak{r}(q)$ is defined for $|q| < 1$ by

$$\mathfrak{r}(q) = \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}}.$$

One of its main properties, due to Rogers [28], is the infinite product formula given by

$$\mathfrak{r}(q) = q^{1/5}\prod_{j=1}^{\infty}(1-q^j)^{\left(\frac{j}{5}\right)} = q^{1/5}\prod_{j=1}^{\infty}\frac{(1-q^{5j-4})(1-q^{5j-1})}{(1-q^{5j-3})(1-q^{5j-2})}$$

where $\left(\frac{j}{p}\right)$ is the Legendre symbol.

This work is about the level 13 analogue defined by

$$R(q) = q\prod_{j=1}^{\infty}(1-q^j)^{\left(\frac{j}{13}\right)}$$
$$= q\prod_{j=1}^{\infty}\frac{(1-q^{13j-12})(1-q^{13j-10})(1-q^{13j-9})(1-q^{13j-4})(1-q^{13j-3})(1-q^{13j-1})}{(1-q^{13j-11})(1-q^{13j-8})(1-q^{13j-7})(1-q^{13j-6})(1-q^{13j-5})(1-q^{13j-2})}.$$

Our goal is to show that although $R(q)$ does not have a simple expansion as a continued fraction, it has many other properties similar to the fifth power of the Rogers–Ramanujan continued fraction. Let us illustrate with two examples.

First, if $r(q) = \mathfrak{r}^5(q)$ then it is well-known that

$$\frac{1}{r(q)} - 11 - r(q) = \frac{1}{q}\prod_{j=1}^{\infty}\frac{(1-q^j)^6}{(1-q^{5j})^6}.$$

Ramanujan found an analogous property for $R(q)$, namely

$$\frac{1}{R(q)} - 3 - R(q) = \frac{1}{q}\prod_{j=1}^{\infty}\frac{(1-q^j)^2}{(1-q^{13j})^2}.$$

This is one of five identities in Entry 8(i) of Chapter 20 in Ramanujan's second notebook [26]. One of these identities is notable for being the last result in the 21 chapters of the notebook to be proved; see the paper by Evans [18] for more information.

Here is the second example. If $r = r(q) = \mathfrak{r}^5(q)$, then it was shown in [9] that

$$q\frac{d}{dq}\log\left(\frac{r}{1-11r-r^2}\right) = \sum_{n=0}^{\infty}a(n)\left(\frac{r(1-11r-r^2)}{(1+r^2)^2}\right)^n \tag{1}$$

where

$$a(n) = \binom{2n}{n}\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{j}.$$