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Casoratian identities for the Wilson and Askey–Wilson polynomials

Satoru Odake[a], Ryu Sasaki[a,b,*]

[a] Department of Physics, Shinshu University, Matsumoto 390-8621, Japan

[b] Center for Theoretical Sciences, National Taiwan University, Taipei 10617, Taiwan

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Abstract

Infinitely many Casoratian identities are derived for the Wilson and Askey–Wilson polynomials in parallel to the Wronskian identities for the Hermite, Laguerre and Jacobi polynomials, which were reported recently by the present authors. These identities form the basis of the equivalence between eigenstate adding and deleting Darboux transformations for solvable (discrete) quantum mechanical systems. Similar identities hold for various reduced form polynomials of the Wilson and Askey–Wilson polynomials, *e.g.* the continuous q-Jacobi, continuous (dual) (q-)Hahn, Meixner–Pollaczek, Al-Salam–Chihara, continuous (big) q-Hermite, etc.

Keywords: Askey–Wilson polynomials; Wilson polynomials; Casoratian identities; Discrete quantum mechanics; Darboux transformations

* Corresponding author at: Department of Physics, Shinshu University, Matsumoto 390-8621, Japan.
E-mail addresses: ryu@yukawa.kyoto-u.ac.jp, sasakiryu@shinshu-u.ac.jp (R. Sasaki).

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1. Introduction

In a previous paper [21] we reported infinitely many Wronskian identities for the Hermite, Laguerre and Jacobi polynomials. They relate the Wronskians of polynomials of *twisted* parameters to the Wronskians of polynomials of *shifted* parameters. Here we will present similar identities for the Wilson and Askey–Wilson polynomials and their reduced form polynomials [1,9,10]. The Wronskians are now replaced by their difference analogues, the Casoratians.

The basic logic of deriving these identities is the same for the Jacobi polynomials etc. and for the Askey–Wilson polynomials etc.; the equivalence between the multiple Darboux–Crum transformations [3,2,11,16,5] in terms of *pseudo virtual state wave functions* and those in terms of *eigenfunctions with shifted parameters*. In other words, the duality between eigenstates adding and deleting transformations. The virtual and pseudo virtual state wave functions have been reported in detail for the differential and difference Schrödinger equations [21,18,22,23,20]. The virtual state wave functions are the essential ingredient for constructing multi-indexed orthogonal polynomials. The pseudo virtual state wave functions play the main role in the above mentioned duality. These Casoratian (Wronskian) identities could be understood as the consequences of the *forward and backward shift relations* and the *discrete symmetries* of the governing Schrödinger equations. The forward and backward shift relations are the characteristic properties of the *classical orthogonal polynomials*, satisfying second order differential and difference equations. These polynomials depend on a set of parameters, to be denoted symbolically by $\boldsymbol{\lambda}$. The forward shift operator $\mathcal{F}(\boldsymbol{\lambda})$ connects $\check{P}_n(x;\boldsymbol{\lambda})$ to $\check{P}_{n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})$, with $\boldsymbol{\delta}$ being the shift of the parameters. For the definition of $\check{P}_n(x;\boldsymbol{\lambda})$, see (2.3) and the paragraph below it. The backward shift operator $\mathcal{B}(\boldsymbol{\lambda})$ connects them in the opposite direction, see (2.18). In the context of quantum mechanical reformulation of the classical orthogonal polynomials [19], the principle underlying the forward and backward shift relations is called *shape invariance* [6].

These identities imply the equality of the deformed potential functions with the twisted and shifted parameters in the difference Schrödinger equations. This in turn guarantees the equivalence of all the other eigenstate wave functions for proper parameter ranges if the self-adjointness of the deformed Hamiltonian and other requirements of quantum mechanical formulation are satisfied. In contrast, the Casoratian identities (3.61)–(3.62), (3.63)–(3.64) are purely algebraic relations and they are valid at generic values of the parameters.

The present work is most closely related in its contents with [23], which formulates deformations of the Wilson and Askey–Wilson polynomials through Casoratians of virtual state wave functions. The relationship of the present work with [23] is the same as that of [21] with [18,22]; derivation of Wronskian–Casoratian identities which reflect the solvability of classical orthogonal polynomials revealed through deformations.

This paper is organised as follows. The formulation of the Wilson and Askey–Wilson polynomials through the difference Schrödinger equations is recapitulated in Section 2. The basic formulas of these polynomials necessary for the present purposes are summarised in Section 2.1. The pseudo virtual states for the Wilson and Askey–Wilson polynomials are introduced and discussed in Section 2.2. Starting with the general properties of the Casoratian determinants in Section 3.1, the eigenstates adding Darboux transformations are recapitulated in Section 3.2. The eigenstates deleting Darboux transformations are summarised in Section 3.3. The Casoratian identities for the Wilson and Askey–Wilson polynomials are presented in Section 3.4. This is the main part of the paper. In Section 4 the Casoratian identities are discussed for the other classical orthogonal polynomials which are obtained by reductions from the Wilson and Askey–Wilson polynomials. The basic formulas of the reduced polynomials are summarised in