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On a quasi-interpolating Bernstein operator☆

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Abstract

We consider a special case of the modification of Lagrange interpolation due to Bernstein. Compared to Lagrange interpolation, these operators interpolate at less points, but they converge for all continuous functions in case of the Chebyshev nodes. Upper and lower estimates for the rate of convergence are given, and the saturation problem is partially solved.

MSC: 41A25; 41A40

Keywords: Bernstein operator; Lagrange interpolation; Chebyshev polynomial; Error estimate; Saturation

1. Introduction

For an $f \in C[-1, 1]$ (the space of continuous functions), consider the Lagrange interpolation

$$L_n(f, x) := \sum_{i=1}^{n} f(x_i)\ell_i(x),$$

where

$$x_i := x_{in} = \cos t_i, \quad t_i := \frac{2i-1}{2n}\pi, \; i = 1, 2, \ldots, n \tag{1}$$

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are the roots of the Chebyshev polynomial $T_n(x) := \cos nt,\ x = \cos t$, and

$$\ell_i(x) := \ell_{in}(x) = \frac{T_n(x)}{T_n'(x_i)(x-x_i)} = \frac{(-1)^{i+1}\cos nt \sin t_i}{n(\cos t - \cos t_i)}, \quad i = 1, 2, \ldots, n$$

are the fundamental polynomials (in this form $x \neq x_i$). It is well-known that Lagrange interpolation cannot be uniformly convergent for all continuous functions, no matter what are the nodes of interpolation. In order to achieve better convergence properties, S.N. Bernstein [1] introduced the following modification of Lagrange interpolation. Let l be a fixed positive integer, $n \geq 2l$, and consider the linear operator

$$B_{n,l}(f,x) := \sum_{i=1}^{m}\sum_{k=1}^{2l-1} f(x_{2(i-1)l+k})\left\{\ell_{2(i-1)l+k}(x) + (-1)^{k+1}\ell_{2il}(x)\right\} + \sum_{k=1}^{n-2lm} f(x_{2lm+k})\ell_{2lm+k}(x), \quad m := \left[\frac{n}{2l}\right], \tag{2}$$

where the last sum (call it the "tail part") contains at most $2l-1$ terms, and when n is a multiple of $2l$ it does not appear at all. In fact, this is a slightly modified form of the original operator of Bernstein (which was established by L. Szili and P. Vértesi [4]). This operator represents a polynomial of degree at most $n-1$, reproduces constants, and has the interpolatory property

$$B_{n,l}(f, x_{2(i-1)l+k}) = f(x_{2(i-1)l+k}), \quad i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, 2l-1$$

and

$$B_{n,l}(f, x_{2lm+k}) = f(x_{2lm+k}), \quad k = 1, 2, \ldots, n-2lm.$$

This means that the interpolation holds at

$$m(2l-1) + n - 2lm = n - m \geq n\left(1 - \frac{1}{2l}\right)$$

points, i.e. by increasing l we increase the number of interpolations. These properties hold for any system of nodes.

The idea in this definition is that the sum or difference of the two fundamental functions of interpolation in the formula (2), in most subintervals determined by adjacent nodes, is always smaller than the fundamental functions themselves. This is how Bernstein was able to prove that

$$\lim_{n\to\infty} \|f - B_{n,l}(f)\| = 0 \tag{3}$$

for all $f \in C[-1,1]$.

Szili and Vértesi [4] considered Jacobi nodes with parameters α, β and proved the point wise error estimate

$$|f(x) - B_{n,l}^{(\alpha,\beta)}(f,x)| \leq c(\alpha,\beta)\sum_{i=1}^{n}\omega\left(f, \frac{i}{n}\sqrt{1-x^2} + \frac{i^2}{n^2}\right)\frac{1}{i^\gamma}, \quad |x| \leq 1, \tag{4}$$

where $B_{n,l}^{(\alpha,\beta)}(f,x)$ is the above operator considered for the Jacobi nodes, $c(\alpha,\beta) > 0$ is independent of n and f, $-1 < \alpha, \beta < 1/2$, $\gamma := \min(2, 3/2-\alpha, 3/2-\beta)$, and $\omega(f,\cdot)$ is the modulus of continuity of $f \in C[-1,1]$.