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Journal of Approximation Theory 196 (2015) 101–110

JOURNAL OF Approximation Theory

www.elsevier.com/locate/jat

Full length article

Norm and smoothness of a function related to the coefficients of its expansion

Z. Ditzian

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Received 24 September 2014; received in revised form 23 February 2015; accepted 2 March 2015
Available online 14 March 2015

Communicated by Feng Dai

Abstract

Relations between the coefficients of expansions of functions on S^{d-1} and on $[-1, 1]$ with the norm, or with the smoothness of those functions are given. Estimates following the Hausdorff–Young and the Hardy–Littlewood-type inequalities are unified, generalized and applied.

MSC: 41A17; 41A25; 42C10

Keywords: Spherical harmonic polynomials; Jacobi weights; Hausdorff–Young and Hardy–Littlewood inequalities

1. Introduction

The object of this paper is to show how the results of Hausdorff–Young-type and those of Hardy–Littlewood-type inequalities can be unified to obtain new more general inequalities containing both of them. As a consequence, following [5,7], we obtain new relations between the coefficients of the expansion and the smoothness of the functions expanded.

In Section 2 we give details of the results for expansions of the function $f(\boldsymbol{x})$ by spherical harmonic polynomials where $\boldsymbol{x} \in S^{d-1}$ and S^{d-1} is the unit sphere in R^d. In Section 3 we give details of the results for expansion of $f(x)$, $x \in [-1, 1]$ with respect to Jacobi-type weights, by polynomials.

E-mail address: zditzian@shaw.ca.

http://dx.doi.org/10.1016/j.jat.2015.03.002

In Section 4 we conclude with some remarks. We also take the opportunity in Sections 2 and 3 to correct some previous errors of mine.

2. Expansion by spherical harmonic polynomials

Following the notation in [7, Section 5], the Laplace–Beltrami operator $\widetilde{\Delta}$ on $S^{d-1} = \{\boldsymbol{x} \in R^d : \|\boldsymbol{x}\| = 1\}$ is the tangential component of the Laplacian Δ. The eigenspaces of $\widetilde{\Delta}$, H_k are given by $H_k \equiv \{\varphi : \widetilde{\Delta}\varphi = -k(k+d-2)\varphi\}$. We set $\{Y_{k,\ell}\}_{\ell=1}^{d_k}$ to be any orthonormal basis of H_k and recall that $d_k \approx (1+k)^{d-2}$ where $b_k \approx c_k$ means that $C^{-1}c_k \le b_k \le Cc_k$. The expansion of f is: $f \sim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} a_{k,\ell} Y_{k,\ell}$ where $a_{k,\ell}$ are given by $a_{k,\ell} = a_{k,\ell}(f) = \int_{S^{d-1}} f(\boldsymbol{x}) \overline{Y}_{k,\ell}(\boldsymbol{x}) d\boldsymbol{x}$. We further set $A_k = A_k(f)$ to be $A_k \equiv \left(\sum_{\ell=1}^{d_k} |a_{k,\ell}|^2\right)^{1/2}$. The unification of the Hausdorff–Young and the Hardy–Littlewood-type inequalities on the unit sphere is given in the following theorem.

Theorem 2.1. *Suppose* $f \in L_p(S^{d-1})$, $1 < p \le 2$, $p^{-1} + q^{-1} = 1$, $p \le s \le q$, $d \ge 3$, $f \sim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} a_{k,\ell} Y_{k,\ell}$ *and* $A_k = \left(\sum_{\ell=1}^{d_k} |a_{k,\ell}|^2\right)^{1/2}$. *Then*

$$\sum_{k=0}^{\infty} \left\{ (1+k)^{\left(\frac{1}{q}-\frac{1}{s}\right)\frac{d}{2} + (d-2)\left(\frac{1}{s}-\frac{1}{p}\right)\frac{1}{2}} A_k \right\}^s \le C \|f\|^s_{L_p(S^{d-1})}. \tag{2.1}$$

Suppose for $p \le r \le q$, $2 \le q < \infty$, $p^{-1} + q^{-1} = 1$ *and* $\left(\sum_{\ell=1}^{d_k} |c_{k,\ell}|^2\right)^{1/2} \equiv C_k$ *the sequence* $\{c_{k,\ell}\}_{k=0,\ell=1}^{\infty,d_k}$ *satisfies the inequality*

$$\sum_{k=0}^{\infty} \left\{ (1+k)^{\left(\frac{1}{p}-\frac{1}{r}\right)\frac{d}{2} + (d-2)\left(\frac{1}{r}-\frac{1}{q}\right)\frac{1}{2}} C_k \right\}^r < \infty. \tag{2.2}$$

Then $f \sim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} c_{k,\ell} Y_{k,\ell}$ *satisfies* $f \in L_q(S^{d-1})$ *with* $c_{k,\ell} = \int_{S^{d-1}} f(\boldsymbol{x}) \overline{Y}_{k,\ell}(\boldsymbol{x}) d\boldsymbol{x}$ *and*

$$\|f\|^r_{L_q(S^{d-1})} \le M \sum_{k=1}^{\infty} \left\{ (1+k)^{\left(\frac{1}{p}-\frac{1}{r}\right)\frac{d}{2} + (d-2)\left(\frac{1}{r}-\frac{1}{q}\right)\frac{1}{2}} C_k \right\}^r. \tag{2.3}$$

We note that the conclusion (2.3) is valid for $f \sim \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} c_{k,\ell} Y_{k,\ell}$ for any orthonormal basis $\{Y_{k,\ell}\}_{\ell=1}^{d_k}$ of H_k.

We observe that in the inequalities (2.1) and (2.3) we need $p > 1$ and $q < \infty$ respectively. This follows from the fact that (2.1) fails when $p = s = 1$ and the analog of (2.3) for $q = r = \infty$ also fails. The same observation can be made for Theorem 3.1 where (3.3) fails for $p = s = 1$ and the analog of (3.5) fails for $q = r = \infty$. These facts will be demonstrated in Theorems 4.1 and 4.2.

Remark 2.2. We note that for $s = p$ and $r = q$ Theorem 2.1 contains Theorem 5.2, (5.7) and (5.9), of [7]. For $s = q$ and $r = p$ Theorem 2.1 corrects an error in Theorem 5.1, (5.4) and (5.5), where the power q should have been on $d_k^{\frac{1}{q}-\frac{1}{2}} = d_k^{(\frac{1}{q}-\frac{1}{p})/2}$ and the power p on $d_k^{(\frac{1}{p}-\frac{1}{2})} = d_k^{(\frac{1}{p}-\frac{1}{q})/2}$ respectively. In fact, using the extreme cases i.e. when $p = q = 2$ and $p = 1$, $q = \infty$ (Parseval identity and ([7], (5.4)′)), the Stein modification of the Riesz–Thorin theorem, which is also given below in Theorem 2.3, implies (and proves) the above and not as