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The b-adic tent transformation for quasi-Monte Carlo integration using digital nets[☆]

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Abstract

In this paper we investigate quasi-Monte Carlo (QMC) integration using digital nets over $\mathbb{Z}_b$ in reproducing kernel Hilbert spaces. The tent transformation (previously called baker's transform) was originally used for lattice rules by Hickernell (2002) to achieve higher order convergence of the integration error for smooth non-periodic integrands, and later, has been successfully applied to digital nets over $\mathbb{Z}_2$ by Cristea et al. (2007) and Goda (2015). The aim of this paper is to generalize the latter two results to digital nets over $\mathbb{Z}_b$ for an arbitrary prime b. For this purpose, we introduce the *b-adic tent transformation* for an arbitrary positive integer b greater than 1, which is a generalization of the original (dyadic) tent transformation. Further, again for an arbitrary positive integer b greater than 1, we analyze the mean square worst-case error of QMC rules using digital nets over $\mathbb{Z}_b$ which are randomly digitally shifted and then folded using the b-adic tent transformation in reproducing kernel Hilbert spaces. Using this result, for a prime b, we prove the existence of good higher order polynomial lattice rules over $\mathbb{Z}_b$ among a smaller number of candidates as compared to the result by Dick and Pillichshammer (2007), which achieve

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almost the optimal convergence rate of the mean square worst-case error in unanchored Sobolev spaces of smoothness of arbitrary high order.

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1. Introduction

In this paper we are interested in approximating multivariate integrals of functions defined over the s-dimensional unit cube

$$I(f) := \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

by quasi-Monte Carlo (QMC) rules

$$Q(f; P_{N,s}) := \frac{1}{N}\sum_{\boldsymbol{x}\in P_{N,s}} f(\boldsymbol{x}),$$

where $P_{N,s} \subset [0, 1]^s$ is a point set consisting of N points. In order to obtain a small integration error, we need to choose $P_{N,s}$ carefully depending on the class of integrands under consideration. Two prominent ways to construct good point sets are integration lattices, see, e.g., [15,18], and digital nets and sequences, see, e.g., [9,15]. QMC rules based on integration lattices are usually called lattice rules. In this paper, we focus on QMC rules using digital nets as point sets.

The typical convergence rate of the integration error using QMC rules is $O(N^{-1+\varepsilon})$ with arbitrarily small $\varepsilon > 0$. In order to achieve higher order convergence of the integration error, it is of interest to study how to construct point sets which can exploit the smoothness of an integrand. It has long been known that it is possible to achieve higher order convergence for smooth periodic integrands by using lattice rules, whereas neither lattice rules nor QMC rules using digital nets can exploit the smoothness of non-periodic integrands so as to achieve higher order convergence. More recently, regarding QMC rules using digital nets, Dick [5,6] analyzed the decay of Walsh coefficients of smooth periodic and non-periodic functions, respectively, and introduced higher order digital nets that can achieve higher order convergence. Higher order polynomial lattice point sets, which were first studied in [8] by generalizing the definition of polynomial lattice point sets in [14], are one of the special examples of higher order digital nets. (In this paper, we shall use the word *digital nets* as a generic term that includes higher order digital nets.) Regarding lattice rules, on the other hand, the tent transformation, also known as the baker's transformation, was used by Hickernell [12] to achieve higher order convergence for non-periodic integrands in unanchored Sobolev spaces of smoothness of second order. Here we note that the tent transformation has been originally introduced and studied in the context of dynamical systems, see, e.g., [16].

The tent transformation was later analyzed in the context of QMC rules using digital nets by Cristea et al. [4], where the tent transformation was successfully applied to randomly digitally shifted digital nets over $\mathbb{Z}_2$ to achieve almost the optimal convergence rate for integrands in unanchored Sobolev spaces of smoothness of second order. Their result has been generalized