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The electrostatic properties of zeros of exceptional Laguerre and Jacobi polynomials and stable interpolation

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Abstract

We examine the electrostatic properties of exceptional and regular zeros of X_m-Laguerre and X_m-Jacobi polynomials. Since there is a close connection between the electrostatic properties of the zeros and the stability of interpolation on the system of zeros, we can deduce an Egerváry–Turán type result as well. The limit of the energy on the regular zeros is also investigated.

MSC: 33E30; 41A05

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1. Introduction

Classical orthogonal polynomials can be introduced as the eigenfunctions of Sturm–Liouville operators and they also play a fundamental role in the construction of bound-state solutions to exactly solvable potentials in quantum mechanics. Moreover an equilibrium problem for the logarithmic interaction of positive unit charges under an external field leads to a nice electrostatic interpretation of their zeros (cf. e.g. [26] or [27]). The following approach of generalization was

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investigated in the 90s: let $w = e^{-Q}$ be a positive weight function supported on $(a, b) \subset \mathbb{R}$. Supposing that w has finite moments, let us introduce the orthogonal polynomials on (a, b) with respect to w: $p_n = p_n(w)$. When Q is twice differentiable and convex, p_n satisfies the following differential equation on (a, b):

$$p_n''(x) + M_n(x)p_n'(x) + N_n(x)p_n(x) = 0.$$

Here $M_n(x)$ and $N_n(x)$ depend on n, actually on p_n (cf. e.g. Theorem 3.4. in [20,16]). The properties of this type of general orthogonal polynomials were examined by several authors. The electrostatic behavior of their zeros was also investigated (cf. [15]). On the other hand these general orthogonal polynomials are not the suitable ones for constructing solvable potentials. Recently some new families of orthogonal polynomials are investigated which are very useful to this purpose. At first X_1-Jacobi and X_1-Laguerre polynomials, as exceptional orthogonal polynomial families were introduced by D. Gómez-Ullate, N. Kamran and R. Milson (cf. e.g. [8,10]). The relationship between exceptional orthogonal polynomials and the Darboux transform is observed by C. Quesne (cf. e.g. [23]). Higher-codimensional families were introduced by S. Odake and R. Sasaki [22]. The location of zeros of exceptional orthogonal polynomials is described by D. Gómez-Ullate, F. Marcellán and R. Milson [12], and the electrostatic interpretation of zeros of X_1-Jacobi polynomials is given by D. Dimitrov and Yen Chi Lun [3].

Below we extend some results of [3] to X_m-Laguerre polynomials of the first kind. Although it is a much larger class than the one discussed in [3], it is far for being the most general class. The exceptional Laguerre and Jacobi polynomials discussed below are reachable from the classical polynomials by 1-step Darboux transformations, but there are much general classes of exceptional orthogonal polynomials. More precisely as it was conjectured in [11] and proved for Hermite subclass in [7], "every X_m orthogonal polynomial system for any codimension m can be obtained by applying a sequence of at most m Darboux transformations to a classical orthogonal polynomial system".

Besides the extension of some earlier results, we show that the regular zeros of X_m-Jacobi and X_m-Laguerre polynomials behave like the zeros of the classical orthogonal polynomials in point of energy. To this purpose we adapt some methods to exceptional orthogonal polynomials, which were developed to the general ones. Since the regular zeros of exceptional polynomials form a minimal energy (or Fekete) system under a suitable external field, similarly to [14] we will show that on these sets one can build up stable interpolation operators which are the most economical as well. Finally the notion of Fekete sets and nth transfinite diameter allows to investigate the behavior of the energy function when the number of the points tends to infinity.

2. Preliminaries

We examine an energy problem under an external field on a finite or an infinite interval of $\mathbb{R}$. To this end we introduce some notations.

Let $U_n := \{u_1, \dots, u_n\}$ be any system of nodes on an interval I and $0 \le w \in C^2(I)$ be a weight function on I. Let $\omega_{U_n}(x) := \prod_{k=1}^{n}(x - u_k)$.

Definition 1. The energy function on I with respect to w is

$$T_w(u_1, \dots, u_n) = \prod_{j=1}^{n} w(u_j) \prod_{1 \le i < j \le n} (u_i - u_j)^2,$$

cf. e.g. p. 143. in [25] or (2.5) in [15].