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Domain of convergence for a series of orthogonal polynomials

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Abstract

Let $\{p_k\}_{k=0}^{\infty}$ be the orthogonal polynomials with certain exponential weights. In this paper, we prove that under certain mild conditions on exponential weights class, a series of the form $\sum b_k p_k$ converges uniformly and absolutely on compact subsets of an open strip in the complex plane, and diverges at every point outside the closure of this strip.

Keywords: Series; Convergence; Orthogonal polynomials; Exponential weights

1. Introduction and main result

Let $\mathbb{R}^+ = [0, \infty)$. A function $g : \mathbb{R}^+ \to (0, \infty)$ is said to be quasi-increasing if there exists $C > 0$ such that $g(x) \le Cg(y)$ for $0 < x \le y < \infty$. Similarly we may define the notation of a quasi-decreasing function. The notation $f(x) \sim g(x)$ means that there are positive constants C_1, C_2 independent of x such that $C_1 \le f(x)/g(x) \le C_2$ for all x. For any positive numbers $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ we define $c_n \sim d_n$. Similarly, the notation is used for sequences of functions. Throughout this paper $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t or

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polynomials $P(x)$ of degree at most n. We write $C = C(\lambda), C \neq C(\lambda)$ to indicate dependence on, or independence of, a parameter λ. The same symbol does not necessarily denote the same constant in different occurrences. We denote the class of polynomials with degree n by $\mathcal{P}_n$.

In this paper, we will consider the following class of exponential weights defined in [2, Definition 1.4].

Definition 1.1. Let $Q : \mathbb{R} \to \mathbb{R}^+$ satisfy the following properties:

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.
(b) $Q'(x)$ is non-decreasing in $\mathbb{R}$.
(c)
$$\lim_{x\to+\infty} Q(x) = \lim_{x\to-\infty} Q(x) = \infty.$$
(d) The function
$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$
is quasi-increasing in $(0, \infty)$ and quasi-decreasing in $(-\infty, 0)$, with
$$T(x) \geq \Lambda > 1, \quad x \in \mathbb{R} \setminus \{0\}.$$
(e) There exists $\varepsilon_0 \in (0, 1)$ such that for $y \in \mathbb{R} \setminus \{0\}$,
$$T(y) \sim T\left(y\left|1 - \frac{\varepsilon_0}{T(y)}\right|\right).$$
(f) Assume that there exist $C, \varepsilon_1 > 0$ such that
$$\int_{x-\frac{\varepsilon_1|x|}{T(x)}}^{x} \frac{|Q'(s) - Q'(x)|}{|s - x|^{3/2}} ds \leq C|Q'(x)|\sqrt{\frac{T(x)}{|x|}}, \quad x \in \mathbb{R} \setminus \{0\}.$$

Then we write $w(x) = \exp(-Q(x)) \in \mathcal{F}(Lip\frac{1}{2})$.

From now on, we let $w = \exp(-Q) \in \mathcal{F}\left(Lip\frac{1}{2}\right)$. Then we can construct the orthonormal polynomials $p_n(x) = p_n(w^2, x)$ of degree n, $n = 0, 1, 2, \ldots$ for $w^2(x)$, that is,
$$\int_{-\infty}^{\infty} p_n(x)p_m(x)w^2(x)dx = \delta_{mn} \quad \text{(Kronecker delta)}.$$

In this paper, we prove that under certain mild conditions on $w = \exp(-Q) \in \mathcal{F}(Lip\frac{1}{2})$, a series of the form $\sum b_k p_k$ converges uniformly and absolutely on compact subsets of an open strip in the complex plane, and diverges at every point outside the closure of this strip.

The numbers $a_{-t} < 0 < a_t, t > 0$ are uniquely determined by the following equations
$$t = \frac{1}{\pi}\int_{a_{-t}}^{a_t} \frac{xQ'(x)}{\sqrt{(x - a_{-t})(a_t - x)}}dx; \tag{1.1}$$
$$0 = \frac{1}{\pi}\int_{a_{-t}}^{a_t} \frac{Q'(x)}{\sqrt{(x - a_{-t})(a_t - x)}}dx.$$

Then we see that a_t is an increasing function of $t \in \mathbb{R}$, with
$$\lim_{t\to-\infty} a_t = -\infty; \qquad \lim_{t\to\infty} a_t = \infty.$$