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Painlevé III asymptotics of Hankel determinants for a singularly perturbed Laguerre weight

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Abstract

In this paper, we consider the Hankel determinants associated with the singularly perturbed Laguerre weight $w(x) = x^{\alpha} e^{-x-t/x}$, $x \in (0, \infty)$, $t > 0$ and $\alpha > 0$. When the matrix size $n \to \infty$, we obtain an asymptotic formula for the Hankel determinants, valid uniformly for $t \in (0, d]$, $d > 0$ fixed. A particular Painlevé III transcendent is involved in the approximation, as well as in the large-n asymptotics of the leading coefficients and recurrence coefficients for the corresponding perturbed Laguerre polynomials. The derivation is based on the asymptotic results in an earlier paper of the authors, obtained by using the Deift–Zhou nonlinear steepest descent method.

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1. Introduction and statement of results

Let $w(x)$ be the following singularly perturbed Laguerre weight

$$w(x) = w(x; t, \alpha) = x^{\alpha} e^{-V_t(x)}, \quad x \in (0, \infty),\ t > 0,\ \alpha > 0 \tag{1.1}$$

with

$$V_t(x) := x + \frac{t}{x}, \quad x \in (0, \infty),\ t > 0. \tag{1.2}$$

In this paper, we consider the Hankel determinant associated with the above weight function

$$D_n[w; t] := \det(\mu_{j+k})_{j,k=0}^{n-1}, \tag{1.3}$$

where μ_j is the jth moment of $w(x)$, i.e.

$$\mu_j := \int_0^{\infty} x^j w(x) dx. \tag{1.4}$$

It is well-known that Hankel determinants are closely related to partition functions in random matrix theory. Indeed, let $Z_n(t)$ be the partition function associated with the weight function in (1.1)

$$Z_n(t) := \int_0^{+\infty} \cdots \int_0^{+\infty} \prod_{1 \le j < k \le n} (\lambda_i - \lambda_j)^2 \lambda_j^{\alpha} \exp\left(-\sum_{j=1}^{n} V_t(\lambda_j)\right) d\lambda_1 \cdots d\lambda_n. \tag{1.5}$$

Then, there is only a constant difference between $D_n[w; t]$ and $Z_n(t)$, that is

$$D_n[w; t] = \frac{1}{n!} Z_n(t); \tag{1.6}$$

for example, see [6, Eq. (1.2)]. When $t = 0$, $w(x)$ is reduced to the classical Laguerre weight. The corresponding partition function $Z_n(0)$ is associated with the Laguerre unitary ensemble and given explicitly below

$$Z_n(0) = \prod_{j=1}^{n} j!\, \Gamma(j + \alpha); \tag{1.7}$$

see [21, p. 321].

The matrix model and Hankel determinants $D_n[w; t]$ associated with the weight function $w(x)$ in (1.1) were first considered by Osipov and Kanzieper [24] in bosonic replica field theories. Later, Chen and Its [6] consider this problem again from another point of view, where their motivation partially originates from an integrable quantum field theory at finite temperature. When n is fixed, they showed that the Hankel determinant $D_n[w; t]$ is the isomonodromy τ-function of a Painlevé III equation multiplied by two factors, that is,

$$D_n[w; t] = C \cdot \tau_n(t)\, e^{\frac{t}{2}}\, t^{\frac{n(n+\alpha)}{2}}, \tag{1.8}$$

where C is a certain constant; see Eq. (6.14) in [6].

In this paper, we focus on the asymptotics of Hankel determinants $D_n[w; t]$ as the matrix size n tends to infinity. In recent years, there has been a considerable amount of interest in the