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Stable regions of Turán expressions

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Abstract

Consider polynomial sequences that satisfy a first-order differential recurrence. We prove that if the recurrence is of a special form, then the Turán expressions for the sequence are weakly Hurwitz stable (non-zero in the open right half-plane). A special case of our theorem settles a problem proposed by S. Fisk that the Turán expressions for the univariate Bell polynomials are weakly Hurwitz stable. We obtain related results for Chebyshev and Hermite polynomials, and propose several extensions involving Laguerre polynomials, Bessel polynomials, and Jensen polynomials associated to a class of real entire functions.

MSC: primary 26D05; secondary 30C10

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1. Introduction

Given a sequence of polynomials $\mathcal{P} = \{P_k\}_{k=0}^{\infty}$, with degree $\deg(P_k) = k$, we adopt the notation

$$\mathscr{T}_k(\mathcal{P}; x) = (P_{k+1}(x))^2 - P_{k+2}(x)P_k(x) \tag{1}$$

for the kth *Turán expression*.

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Let $D = \partial/\partial x$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$ throughout. A polynomial in $\mathbb{C}[x]$ is called *weakly Hurwitz stable* if it is non-zero in the right half-plane $\Re\, x > 0$. Our investigation is motivated, in part, by a problem proposed by S. Fisk.

Problem 1 (*S. Fisk [8, p. 724, Question 33]*)**.** Let $\mathcal{B} = \{B_k(x)\}_{k=0}^{\infty}$, where B_k is the kth univariate Bell polynomial given by the recurrence

$$B_{k+1}(x) = x(B_k(x) + DB_k(x)), \quad k \in \mathbb{N},$$

with initial value $B_0(x) = 1$. Then $\mathscr{T}_k(\mathcal{B}; x)$ is weakly Hurwitz stable for $k \in \mathbb{N}$.

Problem 1 concerns the univariate Bell polynomials, which have significance in combinatorics [2,13], [17, p. 76], and states that an associated Turán expression is weakly Hurwitz stable, a property relevant in the analysis of linear systems [1, p. 465]. Unfortunately, Fisk passed away before he could finish his book, and the tentative proof of Problem 1 which appears in the electronic draft [8, p. 651, Lemma 21.91] is incorrect, as we show in Example 2.5 (this may be why it also appears in the open question list [8, p. 724, Question 33]). By means of the following theorem we are able to answer Problem 1 in the affirmative.

Theorem 1.1. *Let $\mathcal{P} = \{P_k\}_{k=0}^{\infty}$ be a set of real polynomials such that* $\deg(P_k) = k$*, for $k \in \mathbb{N}$*

(i) *If the sequence $\mathcal{P}$ satisfies*

$$P_{k+1}(x) = a(x + b)(D + c_k)P_k(x),$$

$a \neq 0$, $b \geq 0$, and $c_{k+1} \geq c_k > 0$ for all $k \in \mathbb{N}$, then $\mathscr{T}_k(\mathcal{P}; x - b)$ is weakly Hurwitz stable for all $k \in \mathbb{N}$.

(ii) *Let $\mathcal{P} = \{P_k\}_{k=0}^{\infty}$ be a set of polynomials which satisfy*

$$P_{k+1}(x) = c(-ax + b + D)P_k(x)$$

for $a > 0, b, c \in \mathbb{R}$, for each $k \in \mathbb{N}$. For each $k \geq 1$, all k zeros of P_k are real (see Remark 1.2); let M_k denote that largest and m_k denote the smallest zero of P_k. Then for each $k \in \mathbb{N}$, $\mathscr{T}_k(\mathcal{P}; x)$ is non-zero when $\Re\, x > M_{k+1}$ or $\Re\, x < m_{k+1}$.

(iii) *If the sequence $\mathcal{P}$ satisfies*

$$(xD - k)P_k(x) = P_{k-1}(x)$$

for all $k \geq 1$, and each P_k has only non-positive zeros, then $\mathscr{T}_k(\mathcal{P}; x)$ is weakly Hurwitz stable for all $k \in \mathbb{N}$.

Remark 1.2. Note that the types of sequences in Theorem 1.1 parts (i) and (ii) consist of polynomials which have only real zeros. This follows immediately from the recurrences, since the operators $a(x + b)(D + c_k)$ and $c(-ax + b + D)$ preserve the property of having only real zeros (cf. [15, p. 155]). Although real, non-positive zeros are required in the hypothesis in part (iii), the operator $(xD - k)$ preserves the property of having real zeros for polynomials of degree k [15, p. 97]. In addition, in order to avoid a vacuous case for the theorem in parts (i) and (ii) one should take P_0 to be a non-zero real constant.

Indeed, with $a = 1$, $b = 0$, and $c_k = 1$ for all $k \in \mathbb{N}$, Theorem 1.1(i) establishes Problem 1.