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ScienceDirect
Journal of Approximation Theory 192 (2015) 156–192
Journal of Approximation Theory
www.elsevier.com/locate/jat

Full length article

Pointwise approximation with quasi-interpolation by radial basis functions

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Received 24 March 2014; received in revised form 20 August 2014; accepted 11 November 2014
Available online 26 November 2014

Communicated by Joseph Ward

Abstract

We consider radial basis function approximations using at first a localization of the basis functions known as quasi-interpolation (to be contrasted to the plain linear combinations of shifts of radial basis functions or for instance cardinal interpolation). Using these quasi-interpolants we derive various pointwise error estimates in L^p for $p \in [1, \infty)$.

Keywords: Quasi-interpolation; Radial basis functions; Convergence estimates

1. Introduction

Radial basis functions and the linear spaces spanned by their translates provide useful means for multivariate approximations in d-dimensional spaces which are known in a variety of types for at least thirty years [11]. Among the many useful and interesting properties are their approximation orders [5,28] and their ability to adapt to many different applications. In this respect they are similar to other successful approximation methods such as multivariate splines or finite elements [3,8], for instance. Their main forms of approximation are either interpolation at the same points by which they are translated [7], or quasi-interpolation (as detailed below).

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http://dx.doi.org/10.1016/j.jat.2014.11.005

All these approaches have useful aspects and in this article we choose quasi-interpolation [4].

The interesting features and indeed the main advantages of the radial basis function approach are as follows. First and foremost they are available in any dimension of the ambient space, so that it is now possible – in contrast to many of the polynomial spline approaches – to approximate target functions in arbitrary dimensions. In some cases they are therefore the only feasible approach to function approximation. Polynomial splines are, of course, available in multiple dimensions, but may be difficult to construct due to their piecewise structure and the required decompositions of domains. Moreover, many different radial basis functions are available for choices of different smoothness properties, locality (decay or indeed increase) features, possibility of parameters that are built in the radial basis functions and allow individual adjustments of the same basis functions. Finally, existence of the aforementioned interpolants in many cases under mild conditions on the distribution of the data points and convergence properties make these methods attractive.

In this paper, we concentrate to some extend (but not exclusively) on the convergence properties and we find that both for applications and for deriving the convergence properties of the approximants and the spaces where they are from, quasi-interpolation is a suitable ansatz. Pointwise convergence estimates dominating Section 4, the theme of Section 5 is in particular optimal recovery. Section 6 concentrates on simultaneous approximations and least squares. Let us now describe the form of the approximants and give examples for the radial basis functions. Further details and supporting results are to be found in the following two sections.

Radial basis functions typically have the form of translates $\varphi(\|\cdot-\alpha\|)$, where the norm is Euclidean, $\alpha \in \mathbb{R}^d$ are from a fixed, discrete set of points (finite in applications, often infinite in the theoretical analysis, sometimes equally spaced points), and φ is a continuous, univariate function that is called the radial basis function. Typical choices are multiquadrics $\varphi(r) = \sqrt{r^2 + c^2}$ for a positive, fixed parameter c (which may explicitly be zero, in that case we speak about the linear radial basis function $\varphi(r) = r$), or $\varphi(r) = r^2 \log r$, the so-called thin-plate spline [9], or an exponential function $\varphi(r) = \exp(-c^2 r^2)$ or $\varphi(r) = \exp(-cr)$ with the same positive parameter c as above (but here it is not allowed to vanish).

We point out that in particular the last two examples are well-known and highly useful in stochastics because they are positive definite kernel functions and give rise – when used for interpolation on a set of distinct data points – to positive definite interpolation matrices. This is also useful for the numerical solution of the linear interpolation systems, and of course these two radial basis functions are very local (quickly decaying). This locality property is, however, not obvious for the other mentioned radial basis functions, and that is a point where the quasi-interpolation approach helps.

While the idea of interpolation gives obvious conditions on the approximants, namely that the coefficients of the aforementioned translates are defined (if possible) by the matching of the approximant to the approximand at all vectors ξ, the idea of quasi-interpolation is more general. We will specify it in detail at the beginning of the next section, but it should be pointed out already at this stage that since we require locality of the basis functions, most of the examples of radial functions from above as such will not work. We need to form decaying linear combinations from the radial basis functions before we use them for approximation.

There are many ways to form decaying functions as (finite) linear combinations of (increasing) radial basis functions, see for instance [7]; such decaying linear combinations can also stem from local Lagrange functions as used in [12] for a further example. Here we adopt the conditions from [6] as they are general and admit even non-equally spaced data ξ. The resulting localized functions are central to our pointwise-error estimates which are one of the central features of this work. Estimates in L^p-norms were also important, for instance, in the paper [18] about