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Uncertainty principle on weighted spheres, balls and simplexes

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Abstract

For a family of weight functions h_κ that are invariant under a reflection group, the uncertainty principle on the unit sphere in the form of

$$\min_{1\le i\le d}\int_{\mathbb{S}^{d-1}}(1-x_i)|f(x)|^2h_\kappa^2(x)d\sigma\int_{\mathbb{S}^{d-1}}|\nabla_0 f(x)|^2\, h_k^2(x)d\sigma\ge c$$

is established for invariant functions f that have unit norm and zero mean, where ∇_0 is the spherical gradient. In the same spirit, uncertainty principles for weighted spaces on the unit ball and on the standard simplex are established, some of them hold for all admissible functions instead of invariant functions.

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1. Introduction

In the form of the classical Heisenberg inequality, the uncertainty principle in $\mathbb{R}^d$ can be stated as

$$\inf_{a\in\mathbb{R}^d}\int_{\mathbb{R}^d}\|x-a\|^2|f(x)|^2dx\int_{\mathbb{R}^d}|\nabla f(x)|^2dx\ge\frac{d^2}{4}\left(\int_{\mathbb{R}^d}|f(x)|^2dx\right)^2,$$

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where ∇ is the gradient operator. The uncertainty principle has been studied extensively in various settings, see [6,10] and the references therein. Recently, in [2], we established an uncertainty principle on the unit sphere. Let $\mathbb{S}^{d-1}$ denote the unit sphere in $\mathbb{R}^d$ and let ∇_0 denote the spherical gradient on $\mathbb{S}^{d-1}$. The uncertainty inequality in [2] takes the form

$$\min_{e\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} (1-\langle x,e\rangle)|f(x)|^2 d\sigma \int_{\mathbb{S}^{d-1}} |\nabla_0 f(x)|^2\, d\sigma \ge c_d \left(\int_{\mathbb{S}^{d-1}} |f(x)|^2\, d\sigma\right)^2 \tag{1.1}$$

for $f \in L^2(\mathbb{S}^{d-1})$ satisfying $\int_{\mathbb{S}^{d-1}} f(x)d\sigma(x) = 0$, where $d\sigma$ is the surface measure and c_d is a constant depending on the dimension d only. Let $\mathtt{d}(x,y) = \arccos\langle x,y\rangle$ denote the geodesic distance on the sphere. Then $1-\langle x,y\rangle = 2\sin^2 \frac{\mathtt{d}(x,y)}{2} \sim [\mathtt{d}(x,y)]^2$, which shows that the (1.1) is a close analogue of the classical Heisenberg uncertainty principle. Furthermore, the inequality (1.1) is shown to be stronger than an uncertainty inequality previously known in the literature [5,7–9].

The purpose of the present paper is to establish analogues of (1.1) in several different settings. First of all, we consider the weighed space $L^2(h_\kappa^2, \mathbb{S}^{d-1})$ for a family of weight functions h_κ of the form

$$h_\kappa(x) = \prod_{v\in R_+} |\langle x,v\rangle|^{\kappa_v}, \quad \kappa_v \ge 0,$$

that are invariant under a reflection group, where R_+ denotes the set of positive roots that defines the reflection group and κ_v is a nonnegative multiplicity function defined on R_+ whose values are equal whenever reflections in positive roots are conjugate. In the case of the group $\mathbb{Z}_2^d$, the simplest case,

$$h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}, \quad \kappa_i \ge 0.$$

In the setting of a general reflection group, the role of rotation group, under which $d\sigma$ is invariant, is replaced by $h_\kappa^2 d\sigma$ and the partial derivatives are replaced by the Dunkl operators, $\mathcal{D}_1,\ldots,\mathcal{D}_d$, which are first order differential–difference operators that commute with each other [3]. In particular, the gradient is replaced by $\nabla_h := (\mathcal{D}_1,\ldots,\mathcal{D}_d)$ and the operator ∇_0 is replaced by the spherical part of $\nabla_{h,0}$, which coincides with ∇_0 on functions invariant under the reflection group. Secondly, there is a close relation between analysis on the sphere and analysis on the ball, which allows us to consider the setting of $L^2(W,\mathbb{B}^d)$ for a family of weight functions W on the unit ball $\mathbb{B}^d$ of $\mathbb{R}^d$, including the weight function

$$W_{\kappa,\mu}(x) = \prod_{i=1}^d |x_i|^{2\kappa_i}(1-\|x\|^2)^{\mu-1/2}, \quad \kappa_i \ge 0,\ \mu \ge 0,$$

and, in particular, the classical weight function $W_\mu(x) = (1-\|x\|^2)^{\mu-1/2}$ on the ball. Thirdly, a further relation between analysis on $\mathbb{B}^d$ and that on the simplex $\mathbb{T}^d = \{x \in \mathbb{R}^d : x_i \ge 0, 1-x_1-\cdots-x_d \ge 0\}$ allows us to study uncertainty principles on the simplex $\mathbb{T}^d$ with respect to several families of weight functions, including the classical weight functions

$$U_\kappa(x) = x_1^{\kappa_1}\cdots x_d^{\kappa_d}(1-x_1-\cdots-x_d)^{\kappa_{d+1}}, \quad x \in \mathbb{T}^d.$$

Our proof relies on various properties of differential and differential-difference operators on the sphere. The background and the basic results are reviewed in Section 2. Based on the Dunkl