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Multivariate Bernstein operators and redundant systems

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Abstract

The Bernstein operator B_n for a simplex in $\mathbb{R}^d$ is naturally defined via the Bernstein basis obtained from the barycentric coordinates given by its vertices. Here we consider a generalisation of this basis and the Bernstein operator, which is obtained from generalised barycentric coordinates that are given for more general configurations of points in $\mathbb{R}^d$. We call the associated polynomials a Bernstein frame, as they span the polynomials of degree $\leq n$, but may not be a basis. By using this redundant system we are able to give geometrically motivated proofs of some basic properties of the corresponding generalised Bernstein operator, such as the fact it is degree reducing and converges for all polynomials. We also consider the conditions for polynomials in this Bernstein form to join smoothly.

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1. Introduction

The Bernstein operator and its variants have been actively studied for over a century [23,17]. Initially, it was used to give a constructive proof of the Weierstrass density theorem, which culminated in Korovkin's theorem on approximation by positive linear operators [18]. Numerous

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examples have since been given [1], with most being univariate, often with many parameters (akin to the very general families of orthogonal polynomials in the Askey scheme). Over the last forty years, the shape preserving properties of the multivariate Bernstein operator have led to important applications, most notably Bézier curves and surfaces [27,16,15] used in geometric design.

By far the most studied multivariate generalisation is the Bernstein–Durrmeyer operator on a simplex [22,2,9,3,13]. In [24] it was shown that it is not possible to extend the Bernstein operator, and all its properties, to regions which are not simplices (or tensor products of them). Our generalisation is based on a redundant "Bernstein basis", and relaxes the condition of being positive on all of the region.

The Bernstein operator B_n for a simplex in $\mathbb{R}^d$ is defined via the Bernstein basis for $\Pi_n(\mathbb{R}^d)$ (the d-variate polynomials of degree $\leq n$). This basis is obtained by taking powers of the barycentric coordinates given by the vertices of the simplex. In the next section, we outline the basic properties of the affine generalised barycentric coordinates introduced in [29], which are given for more general configurations of points in $\mathbb{R}^d$, e.g., the vertices of a convex polygon. These lead naturally to an analogue of the Bernstein basis, a set of polynomials of degree n which span $\Pi_n(\mathbb{R}^d)$. These are *not* a basis if they are given by more than $d+1$ points, and so we refer to this possibly redundant system as a Bernstein frame (cf. [4]).

In Section 3, we define the generalised Bernstein operator given by a Bernstein frame. We give geometrically motivated proofs of some basic properties of it. These include showing that it is degree reducing and converges for all polynomials, that it reproduces the linear polynomials, and more generally has the same spectral structure as the classical Bernstein operator. Similar arguments in terms of a basis would be far more cumbersome. Finally, we explore some applications of our generalised Bernstein operator. These include a de Casteljau algorithm, shape preservation properties (Section 4), and smoothness conditions in terms of the control points of the associated Bézier surfaces (Section 5).

2. The Bernstein frame

Let V consist of $d+1$ affinely independent points in $\mathbb{R}^d$, i.e., be the vertices of a d-simplex. The **barycentric coordinates** (cf. [10,21]) of a point $x \in \mathbb{R}^d$ with respect to V are the unique coefficients $(\xi_v(x))_{v\in V} \in \mathbb{R}^V$ for which x can be written as an affine combination of the points in V, i.e.,

$$x = \sum_{v\in V} \xi_v(x) v, \qquad \sum_{v\in V} \xi_v(x) = 1. \tag{2.1}$$

We follow [10] and index the barycentric coordinates by the points $v \in V$ that they correspond to, and use standard multi-index notation. It follows, from (2.1), that the ξ_v are linear polynomials which are a basis for $\Pi_1(\mathbb{R}^d)$. More generally, for any $n \geq 1$, the polynomials

$$B_\alpha := \binom{|\alpha|}{\alpha} \xi^\alpha, \quad |\alpha| = n (\alpha \in \mathbb{Z}_+^V)$$

are a basis for $\Pi_n(\mathbb{R}^d)$. Here $|\alpha| = \sum_v \alpha_v$, $\binom{n}{\alpha} = \frac{n!}{\alpha!}$, and $\xi^\alpha = \prod_v \xi_v^{\alpha_v}$.

From now on, let V be a sequence (or multiset) of $m = |V|$ points with affine hull $\mathbb{R}^d$, so that each point $x \in \mathbb{R}^d$ can be written as an affine combination

$$x = \sum_{v\in V} a_v v, \qquad \sum_{v\in V} a_v = 1, \tag{2.2}$$