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# Entropy and approximation numbers of embeddings of weighted Sobolev spaces

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**Abstract**

This paper examines the asymptotic behaviour of entropy and approximation numbers of compact embeddings of weighted Sobolev spaces into Lebesgue spaces. We consider admissible weights multiplied by a polynomial. The main results are related to the distribution of eigenvalues of some degenerate elliptic operators.


*MSC:* 46E35

*Keywords:* Weighted Sobolev spaces; Entropy numbers; Approximation numbers; Degenerate elliptic operators; Continuous and compact embeddings; Admissible functions; Quadratic forms

## 0. Introduction

This paper may be considered as a generalization of [34] where the compactness of the embedding of a certain weighted Sobolev space into a Lebesgue spaces was studied. More precisely, let $B = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$ and $\psi$ be a continuous admissible function on $(0, 1]$ with $\psi(1) = 1$ and bounded from below. Define the weighted Sobolev space

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$E^m_{p,\psi}(B)$, $1 \le p < \infty$, $m \in \mathbb{N}$, as the completion of $C^m_0(B)$ in the norm

$$\|f|E^m_{p,\psi}(B)\| := \left(\int_B |x|^{mp}\, \psi(|x|)^p \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x\right)^{1/p}.$$

We shall prove that, if $\lim_{t\to 0} \psi(t) = \infty$, then the embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$$

is compact, see Proposition 2.5. In Section 2.3 we study the quality of this compactness in terms of the corresponding entropy and approximation numbers of the embedding. In the general setting of $1 \le p < \infty$ we need additional assumptions on the unbounded growth of $\psi$ to get results with standard methods, like $a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim k^{-m/n}$, see Theorem 2.7. But in case of $p = 2$ we will apply specific Hilbert space arguments, what lead to sharp results in Theorem 2.13. In fact, we rely on the spectral theory of degenerate elliptic positive definite self-adjoint operators

$$A^m_\psi f = (-1)^m \sum_{|\alpha|=m} \mathrm{D}^\alpha (b_{m,\psi} \mathrm{D}^\alpha f), \qquad \mathrm{dom}((A^m_\psi)^{1/2}) = E^m_{2,\psi}(B)$$

where $b_{m.\psi}(x) := |x|^{2m}\, \psi^2(|x|)$, see Proposition 2.10. The proof is mainly based on Courant's Max–Min Principle. Nevertheless, the obtained results in Theorems 2.7 and 2.13 follow a principle concerning the asymptotic order of entropy and approximation numbers: the faster $\psi(t)$ tends to infinity as $t \to 0$ the better the compactness. We apply our main results from Theorems 2.7 and 2.13 to example weights of the form

$$\psi(t) = (1 + |\log t|)^\sigma (1 + \log(1 + |\log t|))^\gamma$$

with $\sigma \ge 0$ and $\gamma \in \mathbb{R}$ to indicate the exact influence of $\sigma$ and $\gamma$ on the compactness.

There is a series of papers by Haroske and Skrzypczak [17–19] on compact embeddings of function spaces with Muckenhoupt weights and another series of papers by Kühn, Leopold, Sickel and Skrzypczak [23–26] where the weights are bounded below away from zero and have no singularities. However, in all these papers the spaces were defined on $\mathbb{R}^n$. Also the proof techniques were different, based on wavelet characterizations of the spaces.

## 1. Preliminaries

We fix some notation. By $\mathbb{N}$ we mean the set of natural numbers, by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$, by $\mathbb{Z}^n$ the set of all lattice points in $\mathbb{R}^n$ having integer components, by $\mathbb{R}$ the set of all real numbers and by $\mathbb{R}^n$, $n \in \mathbb{N}$, the Euclidean $n$-space. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in {\mathbb{N}_0}^n$ we fix its length $|\alpha| = \sum_{j=1}^n |\alpha_j|$ and put $\alpha! = \alpha_1! \cdots \alpha_n!$. The derivatives $\mathrm{D}^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n})$ have the usual meaning. Given two (quasi-) Banach spaces $X$ and $Y$, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of $X$ in $Y$ is continuous.

$C^\infty(\mathbb{R}^n)$ collects all infinitely continuously differentiable complex valued functions on $\mathbb{R}^n$ and $C^m_0(\mathbb{R}^n)$ all complex valued compactly supported functions on $\mathbb{R}^n$ having classical derivatives up to order $m \in \mathbb{N}_0$. Furthermore, $u \in L_1^{\mathrm{loc}}([0, \infty))$ means that the complex-valued Lebesgue measurable function $u$ on $\mathbb{R}_+ = (0, \infty)$ is Lebesgue integrable on each interval $(0, a)$ with $a > 0$.

All unimportant positive constants will be denoted by $c$, occasionally with subscripts. For two positive real sequences $\{\alpha_k\}_{k\in\mathbb{N}}$, $\{\beta_k\}_{k\in\mathbb{N}}$ or two positive functions $\phi(x)$, $\psi(x)$ we mean by

$$\alpha_k \sim \beta_k \quad \text{or} \quad \phi(x) \sim \psi(x)$$