Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 192 (2015) 297–305

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Inequalities for Lorentz polynomials

Tamás Erdélyi

Department of Mathematics, Texas A&M University, College Station, TX 77843, United States

Received 9 June 2014; received in revised form 17 December 2014; accepted 29 December 2014
Available online 6 January 2015

Communicated by József Szabados

Abstract

We prove a few interesting inequalities for Lorentz polynomials. A highlight of this paper states that the Markov-type inequality

$$\max_{x\in[-1,1]} |f'(x)| \le n \max_{x\in[-1,1]} |f(x)|$$

holds for all polynomials f of degree at most n with real coefficients for which f' has all its zeros outside the open unit disk. Equality holds only for $f(x) := c((1 \pm x)^n - 2^{n-1})$ with a constant $0 \ne c \in \mathbb{R}$. This should be compared with Erdős's classical result stating that

$$\max_{x\in[-1,1]} |f'(x)| \le \frac{n}{2}\left(\frac{n}{n-1}\right)^{n-1} \max_{x\in[-1,1]} |f(x)|$$

for all polynomials f of degree at most n having all their zeros in $\mathbb{R} \setminus (-1,1)$.

MSC: 11C08; 41A17

Keywords: Lorentz representation of polynomials; Constrained polynomials; Monotone polynomials; Nikolskii-type inequalities; Markov-type inequalities

E-mail address: terdelyi@math.tamu.edu.

http://dx.doi.org/10.1016/j.jat.2014.12.012
0021-9045/

1. Introduction

Let $\mathcal{P}_n$ denote the collection of all polynomials of degree at most n with *real coefficients*. Let $\mathcal{P}_n^c$ denote the collection of all polynomials of degree at most n with *complex coefficients*. Let

$$\|f\|_A := \sup_{x \in A} |f(x)|$$

denote the supremum norm of a complex-valued function f defined on a set A. The Markov inequality asserts that

$$\|f'\|_{[-1,1]} \leq n^2 \|f\|_{[-1,1]}$$

holds for all $f \in \mathcal{P}_n^c$. The inequality

$$|f'(x)| \leq \frac{n}{\sqrt{1-x^2}} \|f\|_{[-1,1]}$$

holds for all $f \in \mathcal{P}_n^c$ and for all $x \in (-1, 1)$, and is known as Bernstein inequality. For proofs of these see [4] or [5], for instance. Various analogues of the above two inequalities are known in which the underlying intervals, the maximum norms, and the family of polynomials are replaced by more general sets, norms, and families of functions, respectively. These inequalities are called Markov-type and Bernstein-type inequalities. If the norms are the same in both sides, the inequality is called "Markov-type", while "Bernstein-type inequality" usually means a pointwise estimate for the derivative. Markov- and Bernstein-type inequalities are known on various regions of the complex plane and the n-dimensional Euclidean space, for various norms such as weighted L_p norms, and for many classes of functions such as polynomials with various constraints, exponential sums of n terms, just to mention a few. Markov- and Bernstein-type inequalities have their own intrinsic interest. In addition, they play a fundamental role in approximation theory.

It had been observed by Bernstein that Markov's inequality for monotone polynomials is not essentially better than for arbitrary polynomials. Bernstein proved that

$$\sup_f \frac{\|f'\|_{[-1,1]}}{\|f\|_{[-1,1]}} = \begin{cases} \dfrac{1}{4}(n+1)^2, & \text{if } n \text{ is odd} \\ \dfrac{1}{4}n(n+2), & \text{if } n \text{ is even,} \end{cases}$$

where the supremum is taken over all $f \in \mathcal{P}_n$ which are monotone on $[-1, 1]$. See [23], for instance. This is surprising, since one would expect that if a polynomial is this far away from having the "equioscillating" property of the Chebyshev polynomial T_n, then there should be a more significant improvement in the Markov inequality. In [16] Erdős gave a class of restricted polynomials for which the Markov factor n^2 improves to cn. He proved that there is an absolute constant c such that

$$|f'(x)| \leq \min \left\{ \frac{c\sqrt{n}}{(1-x^2)^2}, \frac{en}{2} \right\} \|f\|_{[-1,1]}, \qquad x \in (-1, 1),$$

for all $f \in \mathcal{P}_n$ having all their zeros in $\mathbb{R} \setminus (-1, 1)$. This result motivated several people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints. Generalizations of the above Markov- and Bernstein-type inequality of Erdős have been extended in various directions by several people including Lorentz [20], Scheick [24], Szabados [25], Máté [21], P. Borwein [1], Erdélyi [6,7,11–13], Rahman and Schmeisser [23],