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Multiple orthogonal polynomials associated with an exponential cubic weight

Walter Van Assche[a,*], Galina Filipuk[b], Lun Zhang[c]

[a] Department of Mathematics, KU Leuven, Celestijnenlaan 200B box 2400, BE-3001 Leuven, Belgium
[b] Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, Warsaw, 02-097, Poland
[c] School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, People's Republic of China

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Abstract

We consider multiple orthogonal polynomials associated with the exponential cubic weight e^{-x^3} over two contours in the complex plane. We study the basic properties of these polynomials, including the Rodrigues formula and nearest-neighbor recurrence relations. It turns out that the recurrence coefficients are related to a discrete Painlevé equation. The asymptotics of the recurrence coefficients, the ratio of the diagonal multiple orthogonal polynomials and the (scaled) zeros of these polynomials are also investigated.

Keywords: Multiple orthogonal polynomials; Exponential cubic weight; Rodrigues formula; Nearest-neighbor recurrence relations; String equations; Discrete Painlevé equation; Zeros; Asymptotics

* Corresponding author.
E-mail addresses: walter@wis.kuleuven.be (W. Van Assche), filipuk@mimuw.edu.pl (G. Filipuk), lunzhang@fudan.edu.cn (L. Zhang).

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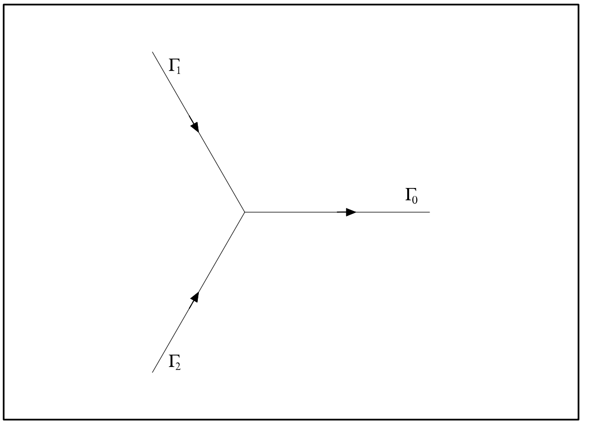

Fig. 1. The three rays $\Gamma_0, \Gamma_1, \Gamma_2$.

1. Introduction and statement of the results

1.1. Orthogonal polynomials associated with an exponential cubic weight

A sequence of non-constant monic polynomials $\{p_n\}$ with $\deg p_n \le n$ is said to be orthogonal with respect to the exponential cubic weight e^{-x^3} if

$$\int_\Gamma p_n(x)x^k e^{-x^3}dx = 0, \quad k = 0, 1, \dots, n-1, \tag{1.1}$$

where the contour Γ is chosen such that the above integral converges. These polynomials satisfy the three-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n^2 p_{n-1}(x), \tag{1.2}$$

where

$$\beta_n = \frac{\int_\Gamma xp_n^2(x)e^{-x^3}dx}{\int_\Gamma p_n^2(x)e^{-x^3}dx}, \qquad \gamma_n^2 = \frac{\int_\Gamma xp_n(x)p_{n-1}(x)e^{-x^3}dx}{\int_\Gamma p_{n-1}^2(x)e^{-x^3}dx}, \tag{1.3}$$

and the initial condition is taken to be $\gamma_0^2 p_{-1} = 0$. It is shown by A. Magnus [19] that the recurrence coefficients β_n and γ_n^2 satisfy the "string" equations

$$\gamma_{n+1}^2 + \beta_n^2 + \gamma_n^2 = 0, \tag{1.4}$$

$$3\gamma_n^2(\beta_{n-1} + \beta_n) = n. \tag{1.5}$$

For the convenience of the reader, we derive the string equations using ladder operators for orthogonal polynomials in the Appendix. Some variants of orthogonal polynomials associated with the exponential cubic weight have recently been studied in the context of numerical analysis [8] and random matrix theory [4].

For our purpose, we are concerned with the polynomials for specific contours Γ. Consider the three rays (see Fig. 1)

$$\Gamma_k = \{z \in \mathbb{C} : \arg z = \omega^k\}, \quad k = 0, 1, 2, \tag{1.6}$$