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Nuttall's theorem with analytic weights on algebraic S-contours

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Dedicated to the memories of Herbert Stahl, brilliant mathematician and a kind friend, and Andrei Alexandrovich Gonchar, great visionary and a wonderful teacher

Abstract

Given a function f holomorphic at infinity, the nth diagonal Padé approximant to f, denoted by $[n/n]_f$, is a rational function of type (n, n) that has the highest order of contact with f at infinity. Nuttall's theorem provides an asymptotic formula for the error of approximation $f - [n/n]_f$ in the case where f is the Cauchy integral of a smooth density with respect to the arcsine distribution on $[-1, 1]$. In this note, Nuttall's theorem is extended to Cauchy integrals of analytic densities on the so-called algebraic S-contours (in the sense of Nuttall and Stahl).

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1. Introduction

Let

$$f(z)=\sum_{k\geq 0} f_k z^{-k} \tag{1}$$

be a convergent power series. A diagonal Padé approximant to f at infinity is a rational function that has the highest order of contact with f at infinity [18,5]. More precisely, let (P_n, Q_n) be a pair of polynomials, each of degree at most n, satisfying

$$R_n(z) := \big(Q_n f - P_n\big)(z) = O\left(1/z^{n+1}\right) \quad \text{as } z\to\infty. \tag{2}$$

It is not hard to verify that the above relation can be equivalently written as a linear system in terms of the Laurent coefficients of f, P_n, and Q_n with one more unknown than equations. Therefore the system is always solvable and no solution of it can be such that $Q_n\equiv 0$ (we may thus assume that Q_n is monic). In general, a solution of (2) is not unique. However, if (P_n, Q_n) and $(\tilde P_n, \tilde Q_n)$ are two distinct solutions, then $P_n\tilde Q_n - \tilde P_n Q_n \equiv 0$ since this difference must behave like $O(1/z)$ near the point at infinity as easily follows from (2). Thus, each solution of (2) is of the form (LP_n, LQ_n), where (P_n, Q_n) is the unique solution of minimal degree. Hereafter, (P_n, Q_n) will always stand for this unique pair of polynomials. A *diagonal Padé approximant* to f of type (n, n), denoted by $[n/n]_f$, is defined as $[n/n]_f := P_n/Q_n$.

We say that a function f of the form (1) belongs to the class $\mathcal{S}$ if it has a meromorphic continuation along any arc originating at infinity that belongs to $\mathbb{C}\setminus E_f$, $\mathrm{cp}(E_f)=0$, and there are points in $\mathbb{C}\setminus E_f$ to which f possesses distinct continuations.[1] Given $f\in\mathcal{S}$, a compact set K is called *admissible* if $\overline{\mathbb{C}}\setminus K$ is connected and f has a meromorphic and single-valued extension there. The following theorems summarize one of the fundamental contributions of Herbert Stahl to complex approximation theory [21–24].

Theorem (*Stahl*). *Given $f\in\mathcal{S}$, there exists the unique admissible compact Δ_f such that $\mathrm{cp}(\Delta_f)\leq\mathrm{cp}(K)$ for any admissible compact K and $\Delta_f\subseteq K$ for any admissible K satisfying $\mathrm{cp}(\Delta_f)=\mathrm{cp}(K)$. Furthermore, Padé approximants $[n/n]_f$ converge to f in logarithmic capacity in $D_f:=\overline{\mathbb{C}}\setminus\Delta_f$. The domain D_f is optimal in the sense that the convergence does not hold in any other domain D such that $D\setminus D_f\neq\varnothing$.*

The minimal capacity set Δ_f, the boundary of the extremal domain D_f, has a rather special structure.

Theorem (*Stahl*). *It holds that*

$$\Delta_f = E_0\cup E_1\cup\bigcup \Delta_j,$$

where $E_0\subseteq E_f$, E_1 consists of isolated points to which f has unrestricted continuations from the point at infinity leading to at least two distinct function elements, and Δ_j are open analytic arcs.

Moreover, the set Δ_f possesses Stahl's symmetry property.

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[1] $\mathrm{cp}(\cdot)$ stands for logarithmic capacity [20].