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Image of Abel–Jacobi map for hyperelliptic genus 3 and 4 curves

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To the memory of Andrei Gonchar and Herbert Stahl

Abstract

The image of Abel–Jacobi embedding of a hyperelliptic Riemann surface of genus less than 5 to its Jacobian is the intersection of several shifted theta divisors with carefully chosen shifts.

Keywords: Riemann surface; Abelian integral; Abel–Jacobi map; (non) special divisor

The necessity for computation of abelian integrals often arises in problems of classical mechanics [11,7], conformal mappings of polygons [4,12,17], solitonic dynamics [1], general relativity [14,8], rational optimization [3,5]—see also [2] and references in the mentioned sources. Function theory on Riemann surfaces allows one to evaluate with computer accuracy such integrals without any quadrature rules. As an example, let us consider Riemann's formula for the

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abelian integral with two simple poles at the points R, Q of the curve $\mathcal{X}$:

$$\eta_{RQ}(P) := \int^{P} d\eta_{RQ} = \log \frac{\theta[\epsilon, \epsilon'](u(P) - u(R), \Pi)}{\theta[\epsilon, \epsilon'](u(P) - u(Q), \Pi)} + \text{const}, \tag{1}$$

where $\theta[\cdot](\cdot, \cdot)$ is Riemann theta function with some odd integer characteristics $[\epsilon, \epsilon']$. It might seem that the usage of this formula still requires the computation of holomorphic abelian integrals involved in the Abel–Jacobi (AJ) map:

$$u(P) := \int_{P_0}^{P} du \in \mathbb{C}^g, \quad du := (du_1, du_2, \ldots, du_g)^t, \tag{2}$$

where g is the genus of the curve $\mathcal{X}$ and du_s are suitably normalized abelian differentials of the first kind.

In this note we show how to avoid the evaluation of AJ map: the image of low genus $g < 5$ hyperelliptic curve in its Jacobian is given as the solution of a (slightly overdetermined) set of equations containing theta functions. Moving along the curve embedded in its Jacobian we can compute the abelian integral by an explicit formula (e.g. (1)) and simultaneously compute a projection of the curve to the complex projective line (see e.g. [10,4,15]). In this way we get a parametric representation [4,12] of abelian integrals which may be used either for the evaluation of integral, or for its inversion. Alternative function theoretic approaches for the computation of abelian integrals use (hyperelliptic) σ and $\wp$ functions [8] or Schottky functions [4, Chapter 6].

Computation of theta function still requires the knowledge of period matrix Π which should be retrieved from the known moduli of the curve. The latter problem is very important, however it is beyond the scope of this note, see e.g. [9] for the expression of period matrix in terms of theta constants.

The techniques developed in this note were used e.g. for the computation of a conformal grid transferred from a rectangle—see Fig. 1 and [13]. The author thanks Oleg Grigoriev for the computations and the anonymous referee for numerous valuable references.

1. Introduction

Let us fix the notations. Consider genus g hyperelliptic curve $\mathcal{X}$:

$$w^2 = \prod_{j=1}^{2g+2} (x - x_j),$$

with distinct branch points $x_1, x_2, \ldots, x_{2g+2}$. The curve admits involution $J(x, w) := (x, -w)$ with fixed points $P_s = (x_s, 0)$, $s = 1, \ldots, 2g + 2$. We introduce a symplectic basis in the homologies of $\mathcal{X}$ as shown in Fig. 2. Dual basis of holomorphic differentials satisfies normalization conditions

$$\int_{a_j} du_s := \delta_{js},$$

and generates the period matrix

$$\int_{b_j} du_s =: \Pi_{js}.$$