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On rational approximation of Markov functions on finite sets

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Dedicated to the memory of Andrei Aleksandrovich Gonchar and Herbert Stahl

Abstract

In this article we study questions related to the approximation of Markov functions on a finite set of points on the real line by rational functions with real coefficients. The main results include formulas for the error in best rational approximation. A connection of rational approximation problems with the discrete Hankel operator is also investigated.

Keywords: Rational approximation; Error in best approximation; Markov function; Hankel operator; Singular values

1. Introduction

1.1. Overview

Let μ be a positive Borel measure with compact support supp $\mu = F$ on the real line $\mathbf{R}$. The Markov function

$$\hat{\mu}(z) = \int_F \frac{d\mu(x)}{z - x} \tag{1}$$

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is the Cauchy transform of the measure μ. We note that the Markov function $\hat{\mu}(z)$ is analytic on $\overline{\mathbf{C}} \setminus F$, where $\overline{\mathbf{C}}$ is the extended complex plane, and $\hat{\mu}(\infty) = 0$. Here and in what follows we assume that F contains infinitely many points (otherwise $\hat{\mu}$ is a rational function) and $F \subset (0, \infty)$.

Let n and m be nonnegative integers. Denote by $\mathcal{P}_n$ the class of all polynomials with real coefficients of degree at most n. Let $\mathcal{R}_{n,m}$ be the class of all rational functions with real coefficients of type (n, m):

$$\mathcal{R}_{n,m} = \{r : r = p/q,\ p \in \mathcal{P}_n,\ q \in \mathcal{P}_m,\ q \not\equiv 0\}.$$

Let f be a real-valued continuous function on a compact set E on the real line $\mathbf{R}$. Denote by $\rho_{n,m}(f; E)$ the error in best rational approximation of the function f in the uniform metric on E by rational functions in the class $\mathcal{R}_{n,m}$:

$$\rho_{n,m}(f; E) = \inf_{r \in \mathcal{R}_{n,m}} \|f - r\|_E,$$

where $\|\cdot\|_E$ is the supremum norm on E.

Rational approximation of Markov functions is a classical topic in the theory of rational approximation of analytic functions. A central place in these investigations belongs to the work of Gonchar [16] in which the degree of rational approximations of Markov functions was analyzed. The Markov theorem [21,5] on the uniform convergence of a diagonal sequence of Padé approximants is one of the basic general results of the theory of convergence of Padé approximants. Questions of convergence of multipoint Padé approximants (interpolation sequences of rational functions with free poles) were studied by Gonchar and Lopez [17], Totik [35], Stahl and Totik [34]. We also mention papers of Anderson [4], Braess [13], Baratchart, Prokhorov and Saff [6–8], Barrett [10], Pekarskii [23] related to rational and meromorphic approximation of Markov functions. In [9] Baratchart, Stahl and Wielonsky gave sharp error rates for the best H_2 approximants (particular Padé approximants) to Markov functions. In the book of Braess [12] some properties of best rational approximants and multipoint Padé approximants for Markov functions are proved, including existence and uniqueness of the corresponding approximants and the absence of defect. We also refer the reader to the book [12] to find results related to approximation by rational functions $r \in \mathcal{R}_{2n-1,2n}$ represented in the form $r = p/q^2$, where $p \in \mathcal{P}_{2n-1}$ and $q \in \mathcal{P}_n$.

In addition to constructive methods of rational approximation, the theory of Hankel operators has been used in recent years to investigate the degree of rational approximation of analytic functions (see [22,30,25]). The methods employed are based on an analysis of the asymptotic behavior of singular values of the Hankel operator constructed from the function being approximated. In a well-known paper [3] Adamyan, Arov and Kreĭn analyzed the singular values of Hankel operators with a continuous operator's symbol on the Hardy space $H_2(G)$ of analytic functions in the unit disk G. The AAK theorem states that the nth singular value of a Hankel operator equals the error in best approximation of a operator's symbol in the space $L_\infty(\partial G)$ by meromorphic functions represented in the form p/q, where p belongs to the Hardy space $H_\infty(G)$ and q is a polynomial of degree at most n (see also [2,29,25]).

In this article we investigate problems of rational approximation of Markov functions on a finite set E_N on the real line using the methods of the theory of Hankel operators. In contrast to classical Hankel operators, that act on the Hardy space $H_2(G)$ of the unit disk G, the discrete Hankel operator acts on polynomials with real coefficients of degree at most n. We establish connections between the smallest singular value of the discrete Hankel operator and the error