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On the zeros of asymptotically extremal polynomial sequences in the plane

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To Herbert Stahl, an exceptional mathematician, a delightful personality, and a dear friend

Abstract

Let E be a compact set of positive logarithmic capacity in the complex plane and let $\{P_n(z)\}_1^\infty$ be a sequence of asymptotically extremal monic polynomials for E in the sense that

$$\limsup_{n\to\infty} \|P_n\|_E^{1/n} \le \mathrm{cap}(E).$$

The purpose of this note is to provide sufficient geometric conditions on E under which the (full) sequence of normalized counting measures of the zeros of $\{P_n\}$ converges in the weak-star topology to the equilibrium measure on E, as $n \to \infty$. Utilizing an argument of Gardiner and Pommerenke dealing with the balayage of measures, we show that this is true, for example, if the interior of the polynomial convex hull of E has a single component and the boundary of this component has an "inward corner" (more generally, a "non-convex singularity"). This simple fact has thus far not been sufficiently emphasized in the literature. As applications we mention improvements of some known results on the distribution of zeros of some special polynomial sequences.

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1. Introduction

Let E be a compact set of positive logarithmic capacity ($\text{cap}(E) > 0$) contained in the complex plane $\mathbb{C}$. We denote by Ω the unbounded component of $\overline{\mathbb{C}} \setminus E$ and by μ_E the *equilibrium measure* (energy minimizing Borel probability measure on E) for the logarithmic potential on E; see e.g. [12, Chapter 3] and [13, Section I.1]. As is well-known, the support $\text{supp}(\mu_E)$ lies on the boundary $\partial\Omega$ of Ω.

For any polynomial $p_n(z)$, of degree n, we denote by ν_{p_n} the *normalized counting measure* for the zeros of $p_n(z)$; that is,

$$\nu_{p_n} := \frac{1}{n} \sum_{p_n(z)=0} \delta_z, \tag{1.1}$$

where δ_z is the unit point mass (Dirac delta) at the point z.

Let $\mathcal{N}$ denote an increasing sequence of positive integers. Then, following [13, p. 169] we say that a sequence of monic polynomials $\{P_n(z)\}_{n\in\mathcal{N}}$, of respective degrees n, is *asymptotically extremal on E* if

$$\limsup_{n\to\infty,\, n\in\mathcal{N}} \|P_n\|_E^{1/n} \le \text{cap}(E), \tag{1.2}$$

where $\|\cdot\|_E$ denotes the uniform norm on E. (We remark that this inequality implies equality for the limit, since $\|P_n\|_E \ge \text{cap}(E)^n$.) Such sequences arise, for example, in the study of polynomials orthogonal with respect to a measure μ belonging to the class **Reg**, see Definition 3.1.2 in [14].

Concerning the asymptotic behavior of the zeros of an asymptotically extremal sequence of polynomials, we recall the following result, see e.g. [10, Theorem 2.3] and [13, Theorem III.4.7].

Theorem 1.1. *Let $\{P_n\}_{n\in\mathcal{N}}$, denote an asymptotically extremal sequence of monic polynomials on E. If μ is any weak-star limit measure of the sequence $\{\nu_{P_n}\}_{n\in\mathcal{N}}$, then μ is a Borel probability measure supported on $\overline{\mathbb{C}} \setminus \Omega$ and $\mu^b = \mu_E$, where μ^b is the balayage of μ out of $\overline{\mathbb{C}} \setminus \Omega$ onto $\partial\Omega$. Similarly, the sequence of balayaged counting measures converges to μ_E:*

$$\nu_{P_n}^b \xrightarrow{*} \mu_E, \quad n \to \infty,\ n \in \mathcal{N}. \tag{1.3}$$

By the weak-star convergence of a sequence of measures λ_n to a measure λ we mean that, for any continuous f with compact support in $\mathbb{C}$, there holds

$$\int f d\lambda_n \to \int f d\lambda, \quad \text{as } n \to \infty.$$

For properties of balayage, see [13, Section II.4].