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Journal of Approximation Theory 189 (2015) 1–28

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Constructing bispectral dual Hahn polynomials

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Received 22 May 2014; accepted 25 September 2014
Available online 8 October 2014

Communicated by Guillermo López Lagomasino

Abstract

Using the concept of $\mathcal{D}$-operator and the classical discrete family of dual Hahn, we construct orthogonal polynomials $(q_n)_n$ which are also eigenfunctions of higher order difference operators.

MSC: 33C45; 33E30; 42C05

Keywords: Orthogonal polynomials; Difference operators and equations; Dual Hahn polynomials; Krall polynomials

1. Introduction

The most important families of orthogonal polynomials are the classical, classical discrete or q-classical families (Askey scheme and its q-analogue). Besides the orthogonality, they are also eigenfunctions of a second order differential, difference or q-difference operator, respectively. That is the case of the dual Hahn polynomials, which are eigenfunctions of a second order difference operator acting in a quadratic lattice.

The issue of orthogonal polynomials which are also eigenfunctions of a higher order differential operator was raised by H.L. Krall in 1939, when he obtained a complete classification for the case of a differential operator of order four [33]. After his pioneer work, orthogonal polynomials which are also eigenfunctions of higher order differential operators are usually called Krall polynomials. This terminology can be extended for finite order difference and q-

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http://dx.doi.org/10.1016/j.jat.2014.09.004
0021-9045/

difference operators. Krall polynomials are also called bispectral, following the terminology introduced by Duistermaat and Grünbaum ([7]; see also [22,23]).

Regarding Krall polynomials, there are important differences depending whether one considers differential or difference operators. Indeed, roughly speaking, one can construct Krall polynomials $q_n(x),\ n \geq 0$, by using the Laguerre $x^\alpha e^{-x}$, or Jacobi weights $(1-x)^\alpha(1+x)^\beta$, assuming that one or two of the parameters α and β are nonnegative integers and adding a linear combination of Dirac deltas and their derivatives at the endpoints of the orthogonality interval (see, for instance, [33,30,29,31,36,37,22,24,25,27,28,45]). This procedure of adding deltas seems not to work if we want to construct Krall discrete polynomials from the classical discrete measures of Charlier, Meixner, Krawtchouk and Hahn (see the papers [3,2] by Bavinck, van Haeringen and Koekoek answering, in the negative, a question posed by R. Askey in 1991 (see p. 418 of [5])).

As it has been shown recently by the author, instead of adding deltas, Krall discrete polynomials can be constructed by multiplying the classical discrete weights by certain polynomials (see [10,15]). The kind of transformation which consists in multiplying a measure μ by a polynomial r is called a Christoffel transform. It has a long tradition in the context of orthogonal polynomials: it goes back a century and a half ago when E.B. Christoffel (see [6] and also [42]) studied it for the particular case $r(x) = x$.

The purpose of this paper is to show that this procedure also works for constructing Krall polynomials from the dual Hahn orthogonal polynomials. Dual-Hahn polynomials $(R_n^{\alpha,\beta,N})_n$ are eigenfunctions of a second order difference operator acting in the quadratic lattice $\lambda(x) = x(x+\alpha+\beta+1)$.

The examples of bispectral dual Hahn polynomials constructed in this paper are also interesting by the following reason. As it has been shown in [12,13], when one applies duality (in the sense of [35]) to Krall–Charlier, Krall–Meixner or Krall–Krawtchouk orthogonal polynomials, exceptional discrete polynomials appear. Exceptional and exceptional discrete orthogonal polynomials p_n, $n \in X \subsetneq \mathbb{N}$, are complete orthogonal polynomial systems with respect to a positive measure which in addition are eigenfunctions of a second order differential or difference operator, respectively. They extend the classical families of Hermite, Laguerre and Jacobi or the classical discrete families of Charlier, Meixner and Hahn.

The last few years have seen a great deal of activity in the area of exceptional orthogonal polynomials (see, for instance, [12,13,20] (where the adjective exceptional for this topic was introduced), [19,21,38,40,39,41], and the references therein). The most apparent difference between classical or classical discrete orthogonal polynomials and their exceptional counterparts is that the exceptional families have gaps in their degrees, in the sense that not all degrees are present in the sequence of polynomials (as it happens with the classical families) although they form a complete orthonormal set of the underlying L^2 space defined by the orthogonalizing positive measure. This means in particular that they are not covered by the hypotheses of Bochner's and Lancaster's classification theorems for classical and classical discrete orthogonal polynomials, respectively (see [4] or [34]).

The connection by duality between Krall discrete and exceptional discrete polynomials is remarkable because anything similar it is known to happen for classical polynomials and differential operators. For exceptional Charlier and Meixner polynomials one can construct exceptional Hermite and Laguerre polynomials by passing to the limit in the same form as one goes from Charlier and Meixner to Hermite and Laguerre in the Askey tableau. The relation between Krall and exceptional polynomials at the level of the classical discrete families is rather helpful even at the classical level. For instance, using it one can lighten the difficult problem of finding necessary and sufficient conditions for the existence of a positive measure with respect to which the excep-