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# On the Lorentz degree of a product of polynomials

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**Abstract**

In this note, we negatively answer two questions of T. Erdélyi (1991, 2010) on possible lower bounds on the Lorentz degree of product of two polynomials. We show that the correctness of one question for degree two polynomials is a direct consequence of a result of Barnard et al. (1991) on polynomials with nonnegative coefficients.


*Keywords:* Polynomials with nonnegative coefficients; Bernstein bases; Lorentz degree; Bézier curves; Degree elevation; Pólya degree

## 1. Introduction

In 1915 S. N. Bernstein [2] observed that a real polynomial $p$ having no zeros in an interval $(a, b)$ admits a representation of the form

$$p(x) = \sum_{k=0}^{m} c_k (x-a)^k (b-x)^{m-k}$$

with all $c_k$ nonnegative or all $c_k$ nonpositive. The smallest integer $m$ for which such representation holds is called the *Lorentz degree* of the polynomial $p$ over the interval $[a, b]$ and denoted by $d_{[a,b]}(p)$. Given two polynomials $p$ and $q$ having no zeros in an interval $(a, b)$, an upper bound on the Lorentz degree of the product $pq$ such as $d_{[a,b]}(pq) \leq d_{[a,b]}(p) + d_{[a,b]}(q)$ can be easily

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derived. However, for possible lower bounds, T. Erdélyi in [5] wrote the following: *As far as I know the following two questions raised in* [4] *are still open. Is it true that*

$$d_{[a,b]}(pq) \geq \min\left(d_{[a,b]}(p), d_{[a,b]}(q)\right) \tag{1}$$

*for any polynomials $p$ and $q$? Is it true that*

$$d_{[a,b]}(pq) \geq |d_{[a,b]}(p) - d_{[a,b]}(q)| \tag{2}$$

*for any polynomials $p$ and $q$?*

In this note, we give counter-examples to the two inequalities (1) and (2). Nevertheless, we show that the first inequality (1) is valid for a large class of real polynomials $p$ and $q$, in particular for degree 2 polynomials. The note is organized as follows: In Section 2, we relate the notion of Lorentz degree to the notion of degree elevation of Bézier curves. The observation of S. N. Bernstein is re-derived using the fact that the sequence of control polygons generated by degree elevation converges to the underlying Bézier curve. In Section 3, we establish a link between the Lorentz degree and the Pólya degree of polynomials. This allows us to use the results of Barnard et al. [1] to prove the first inequality (1) for degree 2 polynomials and a family of polynomials of higher degree. Counter-examples to the two inequalities (1) and (2) are given in Section 4.

## 2. Degree elevation of Bézier curves and Lorentz degree

Throughout this note, $[a, b]$ is a fixed non-trivial real interval. Denote by $\Pi_n$ the linear space of real polynomials of degree at most $n$. Let $p$ be an element of $\Pi_n$ expressed in the Bernstein basis over the interval $[a, b]$ as:

$$p(x) = \sum_{k=0}^{n} p_k B_k^n(x),$$

where

$$B_k^n(x) = \binom{n}{k}\left(\frac{x-a}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k}, \quad k = 0, 1, \ldots, n.$$

For our purpose, it is convenient to represent the graph $\Gamma$ of the polynomial $p$ as the graph of the parametric curve $P(t) = (t, p(t))$ by expressing $P$ in the Bernstein basis over $[a, b]$ as

$$P(t) = \sum_{k=0}^{n} P_k B_k^n(t) \quad \text{with } P_k = \left(\frac{(n-k)a + kb}{n}, p_k\right).$$

The points $P_k$, $k = 0, \ldots, n$ are called the control points of $P$ over the interval $[a, b]$. The curve $\Gamma$ expressed in such a fashion is called a Bézier curve. The polygon $(P_0, P_1, \ldots, P_n)$ is termed the control polygon of the Bézier curve $\Gamma$, see (Fig. 1). Degree elevation is a process that takes a Bézier curve of degree $n$ and expresses it as a Bézier curve of higher degree. For example, the parametric polynomial $P(t) = (t, p(t))$ can be expressed as

$$P(t) = \sum_{k=0}^{n} P_k B_k^n(t) = \sum_{k=0}^{n+1} P_k^{(1)} B_k^{n+1}(t), \quad t \in [a, b].$$

The new control points $P_k^{(1)}$, $k = 0, 1, \ldots, n+1$ can be computed explicitly as [12]:

$$P_0^{(1)} = P_0 \qquad P_{n+1}^{(1)} = P_n \tag{3}$$