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Zeros of polynomials with random coefficients

Igor E. Pritsker*, Aaron M. Yeager

Department of Mathematics, Oklahoma State University, Stilwater, OK 74078, USA

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Abstract

Zeros of many ensembles of polynomials with random coefficients are asymptotically equidistributed near the unit circumference. We give quantitative estimates for such equidistribution in terms of the expected discrepancy and expected number of roots in various sets. This is done for polynomials with coefficients that may be dependent, and need not have identical distributions. We also study random polynomials spanned by various deterministic bases.

Keywords: Polynomials; Random coefficients; Expected number of zeros; Uniform distribution; Random orthogonal polynomials

1. Introduction

Zeros of polynomials of the form $P_n(z) = \sum_{k=0}^{n} A_k z^k$, where $\{A_n\}_{k=0}^{n}$ are random coefficients, have been studied by Bloch and Pólya [4], Littlewood and Offord [18], Erdős and Offord [7], Kac [17], Hammersley [11], Shparo and Shur [22], Arnold [1], and many other authors. The early history of the subject with numerous references is summarized in the books of Bharucha-Reid and Sambandham [3], and of Farahmand [9]. It is now well known that, under mild conditions on the probability distribution of the coefficients, the majority of zeros of these polynomials is accumulating near the unit circumference, and they are also equidistributed in

* Corresponding author.
E-mail addresses: igor@math.okstate.edu (I.E. Pritsker), aaron.yeager@math.okstate.edu (A.M. Yeager).

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the angular sense. Introducing modern terminology, we call a collection of random polynomials $P_n(z) = \sum_{k=0}^{n} A_k z^k,\ n \in \mathbb{N}$, with i.i.d. coefficients, the ensemble of *Kac polynomials*. Let $Z(P_n) = \{Z_1, Z_2, \ldots, Z_n\}$ be the set of complex zeros of a polynomial P_n of degree n. These zeros $\{Z_k\}_{k=1}^{n}$ give rise to the *zero counting measure*

$$\tau_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{Z_k},$$

which is a random unit Borel measure in $\mathbb{C}$. The fact of equidistribution for the zeros of random polynomials can now be expressed via the convergence of τ_n in the weak* topology to the normalized arclength measure $\mu_{\mathbb{T}}$ on the unit circumference $\mathbb{T}$, where $d\mu_{\mathbb{T}}(e^{it}) := dt/(2\pi)$. Namely, we have that $\tau_n \stackrel{*}{\to} \mu_{\mathbb{T}}$ with probability 1 (abbreviated as a.s. or almost surely). More recent papers on zeros of random polynomials include Hughes and Nikeghbali [12], Ibragimov and Zeitouni [14], Ibragimov and Zaporozhets [13], and Kabluchko and Zaporozhets [15,16]. In particular, Ibragimov and Zaporozhets [13] proved that if the coefficients are independent and identically distributed, then the condition $\mathbb{E}[\log^+ |A_0|] < \infty$ is necessary and sufficient for $\tau_n \stackrel{*}{\to} \mu_{\mathbb{T}}$ almost surely. Here, $\mathbb{E}[X]$ denotes the expectation of a random variable X. Furthermore, Ibragimov and Zeitouni [14] obtained asymptotic results on the expected number of zeros when random coefficients are from the domain of attraction of a stable law, generalizing earlier results of Shepp and Vanderbei [20] for Gaussian coefficients.

Our goal is to provide estimates on the expected rate of convergence of τ_n to $\mu_{\mathbb{T}}$. A standard way to study the deviation of τ_n from $\mu_{\mathbb{T}}$ is to consider the discrepancy of these measures in the annular sectors of the form

$$A_r(\alpha, \beta) = \{z \in \mathbb{C} : r < |z| < 1/r,\ \alpha \le \arg z < \beta\}, \quad 0 < r < 1.$$

Such estimates were recently provided by Pritsker and Sola [19] using the largest order statistic $Y_n = \max_{k=0,\ldots,n} |A_k|$. The results of [19] require the coefficients $\{A_k\}_{k=0}^{n}$ be independent and identically distributed (i.i.d.) complex random variables having absolutely continuous distribution with respect to the area measure. This assumption excluded many important distributions such as discrete ones, in particular. We remove many unnecessary restrictions in this paper, and generalize the results of [19] in several directions. Section 2 develops essentially the same theory as in [19] (but uses a different approach) for the case of coefficients that are neither independent nor identically distributed, and whose distributions only satisfy certain uniform bounds for the fractional and logarithmic moments. We also consider random polynomials spanned by general bases in Section 3, which includes random orthogonal polynomials on the unit circle and the unit disk. Section 4 shows how one can handle the discrete random coefficients by methods involving the highest order statistic Y_n, augmenting the ideas of [19]. We further develop the highest order statistic approach to the case of dependent coefficients in Section 5, under the assumption that the coefficients satisfy uniform bounds on the first two moments. All proofs are contained in Section 6.

2. Expected number of zeros of random polynomials

Let $A_k,\ k = 0, 1, 2, \ldots$, be complex valued random variables that are not necessarily independent nor identically distributed, and let $\|P_n\|_\infty = \sup_{\mathbb{T}} |P_n|$, where $\mathbb{T} := \{z : |z| = 1\}$. We study the expected deviation of the normalized number of zeros from $\mu_{\mathbb{T}}$ in annular sectors, which is often referred to as discrepancy between those measures.