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Phase space localization of orthonormal sequences in $L^2_\alpha(\mathbb{R}_+)$

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Abstract

The aim of this paper is to prove a quantitative extension of Shapiro's result on the time–frequency concentration of orthonormal sequences in $L^2_\alpha(\mathbb{R}_+)$. More precisely, we prove that, if $\{\varphi_n\}_{n=0}^{+\infty}$ is an orthonormal sequence in $L^2_\alpha(\mathbb{R}_+)$, then for all $N \geq 0$

$$\sum_{n=0}^{N}\left(\left\|x\varphi_n\right\|^2_{L^2_\alpha} + \left\|\xi\mathcal{H}_\alpha(\varphi_n)\right\|^2_{L^2_\alpha}\right) \geq 2(N+1)(N+1+\alpha),$$

and the equality is attained for the sequence of Laguerre functions. Particularly if the elements of an orthonormal sequence and their Fourier–Bessel transforms (or Hankel transforms) have uniformly bounded dispersions then the sequence is finite.

Moreover we prove the following strong uncertainty principle for bases for $L^2_\alpha(\mathbb{R}_+)$, that is if $\{\varphi_n\}_{n=0}^{+\infty}$ is an orthonormal basis for $L^2_\alpha(\mathbb{R}_+)$ and $s > 0$, then

$$\sup_n\left(\left\|x^s\varphi_n\right\|^2_{L^2_\alpha}\left\|\xi^s\mathcal{H}_\alpha(\varphi_n)\right\|^2_{L^2_\alpha}\right) = +\infty.$$

MSC: 42A68; 42C20

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1. Introduction

A Fourier uncertainty principle is an inequality or uniqueness theorem concerning the joint localization of a function and its Fourier transform. The most familiar form is the Heisenberg–Pauli–Weil inequality. To be more precise, let $d \geq 1$ be the dimension, and let us denote by $\langle \cdot, \cdot \rangle$ the scalar product and by $| \cdot |$ the Euclidean norm on $\mathbb{R}^d$. Then the Heisenberg–Pauli–Weil inequality (see *e.g.* [9,20]) leads to the following classical formulation of the uncertainty principle in form of the lower bound of the product of the dispersions of a unit-norm function in $L^2(\mathbb{R}^d)$ and its Fourier transform:

$$\| |x| f \|_{L^2(\mathbb{R}^d)} \, \| |\xi| \mathcal{F}(f) \|_{L^2(\mathbb{R}^d)} \geq \frac{d}{2}, \tag{1.1}$$

with equality if and only if f is a multiple of a suitable Gaussian. Heisenberg's inequality (1.1) may be also written in the form

$$\| |x| f \|^2_{L^2(\mathbb{R}^d)} + \| |\xi| \mathcal{F}(f) \|^2_{L^2(\mathbb{R}^d)} \geq d, \tag{1.2}$$

where the Fourier transform is defined for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by:

$$\mathcal{F}(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\langle x, \xi \rangle} \, \mathrm{d}x,$$

and it is extended from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ in the usual way. With this normalization, if $f(x) = \tilde{f}(|x|)$ is a radial function on $\mathbb{R}^d$, then $\mathcal{F}(f)(\xi) = \mathcal{H}_{d/2-1}(\tilde{f})(|\xi|)$, where for $\alpha > -1/2$, $\mathcal{H}_\alpha$ is the Fourier–Bessel transform (also known as the Hankel transform) defined by (see *e.g.* [23]):

$$\mathcal{H}_\alpha(\xi) = \int_{\mathbb{R}_+} f(x) j_\alpha(x\xi) \, \mathrm{d}\mu_\alpha(x), \quad \xi \in \mathbb{R}_+ = [0, +\infty).$$

Here $\mathrm{d}\mu_\alpha(x) = \frac{x^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} \, \mathrm{d}x$ and j_α (see *e.g.* [23,25]) is the spherical Bessel function given by:

$$j_\alpha(x) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(x)}{x^\alpha} := \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+\alpha+1)} \left(\frac{x}{2} \right)^{2n}.$$

Note that J_α is the Bessel function of the first kind and Γ is the gamma function.

For $\alpha > -1/2$, let us recall the *Poisson representation formula* (see *e.g.* [24, (1.71.6), p. 15]):

$$j_\alpha(x) = \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} (1-s^2)^{\alpha-1/2} \cos(sx) \, \mathrm{d}s.$$

Therefore, j_α is bounded with $|j_\alpha(x)| \leq j_\alpha(0) = 1$. As a consequence,

$$\|\mathcal{H}_\alpha(f)\|_\infty \leq \|f\|_{L^1_\alpha}, \tag{1.3}$$

where $\|\cdot\|_\infty$ is the usual essential supremum norm and for $1 \leq p < +\infty$, we denote by $L^p_\alpha(\mathbb{R}_+)$ the Banach space consisting of measurable functions f on $\mathbb{R}_+$ equipped with the norms:

$$\|f\|_{L^p_\alpha} = \left(\int_{\mathbb{R}_+} |f(x)|^p \, \mathrm{d}\mu_\alpha(x) \right)^{1/p}.$$