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Recovery of bivariate band-limited functions using scattered translates of the Poisson kernel

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Abstract

This paper continues the study of interpolation operators on scattered data. We introduce the Poisson interpolation operator and prove various properties of this operator. The main result concerns functions whose Fourier transforms are concentrated near the origin, specifically functions belonging to the Paley–Wiener space PW_{B_β}. We show that one may recover these functions from their samples on a complete interpolating sequence for $[-\delta, \delta]^2$ by using the Poisson interpolation operator, provided that $0 < \beta < (3 - \sqrt{8})\delta$.

Keywords: Multivariate interpolation; Paley–Wiener functions; Scattered data

1. Introduction

This paper continues the study of interpolation on scattered data. The basic problem is the following.

Problem. Given a set of sampling nodes $X = \{x_j\} \subset \mathbb{R}^n$ and data $Y = \{y_j\} \subset \mathbb{R}$, find a function L, which satisfies $L(x_j) = y_j$.

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Answers depend on the geometry of sampling nodes, properties of the data, and desired properties of the solution L, e.g. continuity, integrability, etc. We are interested in the case that the sampling nodes X are a complete interpolating sequence for $[-\delta, \delta]^2$, and the data satisfies $Y \in l^2$. Similar hypotheses have been studied before especially in the univariate case. In fact, Lyubarskii and Madych showed in [6] that one may use tempered splines to solve this problem, while Schlumprecht and Sivakumar showed in [8] that the same is true using scattered translates of the Gaussian. Along the same lines, the author showed in [5] that one may use scattered translates of 'regular interpolators' to solve this problem. In particular, one may use the Poisson kernel. In addition to solving the interpolation problem, all of these schemes have the property that a limiting parameter allows one to recover functions from the Paley–Wiener space, however the techniques used do not readily extend to the multivariate case.

Bailey, Schlumprecht, and Sivakumar proved a multivariate extension in [2] by using scattered translates of the Gaussian. Since both the Gaussian and Poisson kernel are regular interpolators, one may expect that a similar multivariate extension can be made for the Poisson kernel as well. We prove such an extension in this paper for the bivariate case. Interestingly, using the Poisson kernel allows us more flexibility in the geometry of the sampling nodes. Specifically, we may use complete interpolating sequences for squares, which have several accessible examples such as the multivariate extension of Kadec's $1/4$-theorem found in [3].

Our main results involve a bound for the Poisson interpolation operator given in Theorem 1, an L^2 recovery result given in Theorem 2, and a pointwise recovery result given in Theorem 3.

The remainder of this paper is organized into four sections. In Section 2, we present preliminary definitions and facts which will be used throughout the paper. The section ends with some examples of complete interpolating sequences. Section 3 introduces the Poisson interpolation operator and ends with our first theorem concerning a bound for this operator. Section 4 contains the statements and proofs of our main recovery results. Finally, an example and additional comments are collected in Section 5 along with a theorem pertaining to $\mathbb{R}^n$.

2. Preliminaries

We begin by adopting the following convention for the Fourier transform.

Definition 1. Let $n \in \mathbb{N}$ and suppose that $g \in L^1(\mathbb{R}^n)$, then the *Fourier transform* of g is denoted $\mathcal{F}[g]$ and given by

$$\mathcal{F}[g](\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} g(x) e^{-i\langle \xi, x\rangle} dx,$$

where $\langle u, v\rangle = \sum_{i=1}^n u_i v_i$ is the standard inner product on $\mathbb{R}^n$.

This convention extends to an $L^2(\mathbb{R}^n)$ isometry.

We are interested in the case that $n = 2$, and make note of the following Fourier transform pair that will be used throughout the remainder of the paper. For $\alpha > 0$ we define the *Poisson kernel* $g_\alpha : \mathbb{R}^2 \to \mathbb{R}$ by

$$g_\alpha(x) = \alpha/(2\pi) \left(\alpha^2 + \langle x, x\rangle\right)^{-3/2},$$

whose Fourier transform is given by

$$\mathcal{F}[g_\alpha](\xi) = e^{-\alpha|\xi|},$$

where $|u| = \sqrt{\langle u, u\rangle}$. There is a corresponding formula for $\mathbb{R}^n$, but we will not need it.