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Approximations of Analytic Functions via Generalized Power Product Expansions

H. Gingold[a], Jocelyn Quaintance[b,*]

[a] West Virginia University, Department of Mathematics, Morgantown WV 26506, USA
[b] Rutgers University, Department of Mathematics, Piscataway NJ 08854, USA

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Abstract

Let $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ be a formal power series. For a fixed set of nonzero complex numbers $\{r_k\}_{k=1}^{\infty}$ we convert $f(x)$ into the formal product $\prod_{k=1}^{\infty}(1+g_k x^k)^{r_k}$, namely the *Generalized Power Product Expansion*. We provide estimates on the domain of absolute convergence of the infinite product when $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ is absolutely convergent. This makes it possible to use the truncated Generalized Power Product Expansions $\prod_{n=1}^{M}(1+g_k x^k)^{r_k}$ as approximations to the analytic function $f(x)$. The results are made possible by certain intriguing algebraic properties characteristic of the expansions for the case of $r_k \geq 1$. An asymptotic formula for the g_k associated with the majorizing power series is provided. A combinatorial interpretation of the *Generalized Power Product Expansion* with $\{r_k\}_{k=1}^{\infty}$ being integers is also given.

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* Corresponding author.
E-mail addresses: gingold@math.wvu.edu (H. Gingold), quaintan@math.rutgers.edu (J. Quaintance).

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1. Introduction

Consider an infinite power series $1+\sum_{n=0}^{\infty} a_n x^n$ that when convergent represents the complex valued function $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$. The subject of this paper is the conversion of the power series with complex coefficients into an infinite product $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n = (1 + g_1 x)^{r_1}(1 + g_2 x^2)^{r_2} \cdots (1 + g_k x^k)^{r_k} \cdots$, where $\{r_k\}$ is an arbitrary set of nonzero complex numbers. The expression $\prod_{k=1}^{\infty}(1 + g_k x^k)^{r_k}$ is the *Generalized Power Product Expansion*, denoted GPPE, and provides a factorization of $f(x)$. Finite truncations of the GPPE, $\left\{\prod_{k=1}^{M}(1 + g_k x^k)^{r_k}\right\}_{M=1}^{\infty}$, provide polynomial approximations for $f(x)$.

Special cases of GPPE appear throughout the literature. The infinite product with elementary factors $(1+x^k)$ is used as a generating function to derive the coefficients $q(n)$ in the power series

$$\prod_{k=1}^{\infty}(1 + x^k) = 1 + \sum_{n=1}^{\infty} q(n)x^n. \tag{1.1}$$

It is the special case of $\prod_{k=1}^{\infty}(1 + g_k x^k)^{r_k}$ with $g_k = 1$ and $r_k = 1$. Equally important is the infinite product known to Euler and his successors,

$$\prod_{k=1}^{\infty}(1 - x^k)^{-1} = 1 + \sum_{n=1}^{\infty} p(n)x^n. \tag{1.2}$$

Each $p(n)$ counts the partitions of a given nonnegative integer into unrestricted parts [2,14]. Eq. (1.2) is $\prod_{k=1}^{\infty}(1 + g_k x^k)^{r_k}$ with $g_k = -1$ and $r_k = -1$.

As these two classical examples suggest, the expansion of a general infinite product $\prod_{k=1}^{\infty}(1+g_k x^k)^{r_k}$ into a power series $1+\Sigma_{n=1}^{\infty} a_n x^n$ generates an infinite sequence of coefficients a_n that count the number of arrangements in a variety of combinatorial configurations. Various combinatorial interpretations for the case of $r_k = 1$ are discussed in [11].

The convergence properties of $\prod_{k=1}^{\infty}(1 + g_k x^k)^{r_k}$ and its companion power series are very important for similar reasons [15,25]. They are crucial in determining the order of growth of the coefficients $a_n = q(n), p(n)$, as n goes to infinity. See [2] for older results and [12,23,22] for contemporary work.

In the twentieth century analysts investigated the convergence of the infinite product $\prod_{k=1}^{\infty}(1 + g_k x^k)$ obtained from the power series $f(x) = 1 + \Sigma_{n=1}^{\infty} a_n x^n$, $a_n \in \mathbb{C}$. The expression $\prod_{k=1}^{\infty}(1 + g_k x^k)$ is called the *Power Product Expansion* of $f(x)$. Working independently several mathematicians developed expressions for the coefficients g_k in terms of the coefficients $\{a_n\}_{n=1}^{\infty}$. The earliest systematic approach goes back to Ritt [25]. Significant work was also done by Borofsky, Feld, Hertzog, Ketchum, Kolberg, A. Knopfmacher, Indlekofer, and Warlimont [5–8,16,18,24,20,19,11,9,12,13,17,21]. Most of these works focused primarily on estimates of the radius of convergence of the Power Product Expansion. The typical result being an estimate of the radius of convergence in terms of the radius of convergence found via logarithmic derivative $\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} d_n z^{n-1}$. The sharpest result in this direction was obtained independently in [17,21]. It stated that the Power Product Expansion converges for $|z| < [\sup |d_k|^{\frac{1}{k}}]^{-1}$, the supremum being taken over all positive integers k.

There are advantages and insights gained by a logarithmic derivative. However, they come with a penalty. We get an indirect expression for the coefficients g_k in terms of the coefficients $d_1, d_2, \ldots, d_k$ rather than a direct expression of g_k in terms of the $a_1, a_2, \ldots, a_k$. Consequently,