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Renorming spaces with greedy bases

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Abstract

We study the problem of improving the greedy constant or the democracy constant of a basis of a Banach space by renorming. We prove that every Banach space with a greedy basis can be renormed, for a given $\varepsilon > 0$, so that the basis becomes $(1+\varepsilon)$-democratic, and hence $(2+\varepsilon)$-greedy, with respect to the new norm. If in addition the basis is bidemocratic, then there is a renorming so that in the new norm the basis is $(1+\varepsilon)$-greedy. We also prove that in the latter result the additional assumption of the basis being bidemocratic can be removed for a large class of bases. Applications include the Haar systems in $L_p[0, 1]$, $1 < p < \infty$, and in dyadic Hardy space H_1, as well as the unit vector basis of Tsirelson space.

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0. Introduction

In approximation theory one is often faced with the following problem. We start with a signal, *i.e.,* a vector x in some Banach space X. We then consider the (unique) expansion $\sum_{i=1}^{\infty} x_i e_i$ of x with respect to some (Schauder) basis (e_i) of X. For example, this may be a Fourier expansion of x, or it may be a wavelet expansion in L_p. We then wish to approximate x by considering m-term approximations with respect to the basis. The smallest error is given by

$$\sigma_m(x) = \inf\left\{ \left\| x - \sum_{i \in A} a_i e_i \right\| : A \subset \mathbb{N},\ |A| \le m,\ (a_i)_{i \in A} \subset \mathbb{R} \right\}.$$

We are interested in algorithms that are easy to implement and that produce the best m-term approximation, or at least get close to it. A very natural process is the greedy algorithm which we now describe. For each $x = \sum x_i e_i \in X$ we fix a permutation $\rho = \rho_x$ of $\mathbb{N}$ (not necessarily unique) such that $|x_{\rho(1)}| \ge |x_{\rho(2)}| \ge \cdots$. We then define the *mth greedy approximant to x*by

$$\mathcal{G}_m(x) = \sum_{i=1}^{m} x_{\rho(i)} e_{\rho(i)}.$$

For this to make sense we need $\inf \|e_i\| > 0$, otherwise (x_i) may be unbounded. In fact, since we will be dealing with democratic bases, all our bases will be *seminormalized*, which means that $0 < \inf \|e_i\| \le \sup \|e_i\| < \infty$. It follows that the biorthogonal functionals (e_i^*) are also seminormalized. Note that a space with a seminormalized basis (e_i) can be easily renormed to make (e_i) *normalized*, *i.e.,* $\|e_i\| = 1$ for all $i \in \mathbb{N}$.

We measure the efficiency of the greedy algorithm by comparing it to the best m-term approximation. We say that (e_i) is a *greedy basis* for X if there exists $C > 0$ (C*-greedy*) such that

$$\|x - \mathcal{G}_m(x)\| \le C\sigma_m(x) \quad \text{for all } x \in X \text{ and for all } m \in \mathbb{N}.$$

The smallest C is the *greedy constant* of the basis. Note that being a greedy basis is a strong property. It implies in particular the strictly weaker property that $\mathcal{G}_m(x)$ converges to x for all $x \in X$. If this weaker property holds, then we say that the basis (e_i) is *quasi-greedy*. This is still a non-trivial property: a Schauder basis need not be quasi-greedy in general.

The simplest examples of greedy bases include the unit vector basis of ℓ_p $(1 \le p < \infty)$ or c_0, or orthonormal bases of a separable Hilbert space. An important and non-trivial example is the Haar basis of $L_p[0, 1]$ $(1 < p < \infty)$ which was shown to be greedy by V.N. Temlyakov [8]. This result was later established by P. Wojtaszczyk [9] using a different method which extended to the Haar system in one-dimensional dyadic Hardy space $H_p(\mathbb{R})$, $0 < p \le 1$. We also mention two recent results. S.J. Dilworth, D. Freeman, E. Odell and Th. Schlumprecht [3] proved that $\left(\oplus_{n=1}^{\infty} \ell_p^n\right)_{\ell_q}$ has a greedy basis whenever $1 \le p \le \infty$ and $1 < q < \infty$. Answering a question raised in [3], G. Schechtman showed that none of the space $\left(\bigoplus_{n=1}^{\infty} \ell_p\right)_{\ell_q}$, $1 \le p \ne q < \infty$, $\left(\bigoplus_{n=1}^{\infty} \ell_p\right)_{c_0}$, $1 \le p < \infty$, and $\left(\bigoplus_{n=1}^{\infty} c_0\right)_{\ell_q}$, $1 \le q < \infty$, have greedy bases [7].

Greedy bases are closely related to unconditional bases. We recall that a basis (e_i) of a Banach space X is said to be *unconditional* if there is a constant K (K*-unconditional*) such that

$$\left\| \sum a_i e_i \right\| \le K \cdot \left\| \sum b_i e_i \right\| \quad \text{whenever } |a_i| \le |b_i| \text{ for all } i \in \mathbb{N}.$$

The best constant K is the *unconditional constant* of the basis which we denote by K_U. The property of being unconditional is easily seen to be equivalent to that of being *suppression*